MTH 101-2020

Assignment 1 : Real Numbers, Sequences

1. Find the supremum of the set $\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\}$.

2. Let A be a non-empty subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$. Show that $\alpha = supA$ if and only if $\alpha - \frac{1}{n}$ is not an upper bound of A but $\alpha + \frac{1}{n}$ is an upper bound of A for every $n \in \mathbb{N}$.

3. Let $y \in (1, \infty)$ and $x \in (0, 1)$. Evaluate $\lim_{n\to\infty}(2n)^y x^n$.

4. For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \frac{1}{4}(x_n^2 + 3)$ for all $n \in \mathbb{N}$. Show that (x_n) converges if and only if $|a| \leq 3$. Moreover, find the limit of the sequence when it converges.

5. Show that the sequence (x_n) defined by $x_1 = \frac{1}{2}$ and $x_{n+1} = \frac{1}{7}\left(x_n^3 + 2\right)$ for $n \in \mathbb{N}$ satisfies the Cauchy criterion.

6. Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ for $n \in \mathbb{N}$. Show that $|x_{2n} - x_n| \geq \frac{1}{2}$ for every $n \in \mathbb{N}$. Does the sequence (x_n) satisfy the Cauchy criterion ?

7. Let (x_n) be defined by $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{x_n + x_{n+1}}{2}$ for $n \geq 1$. Show that (x_n) converges. Further, by observing that $x_{n+2} + \frac{x_{n+1}}{2} = x_{n+1} + \frac{x_n}{2}$, find the limit of (x_n).

Assignment 2 : Continuity, Existence of minimum, Intermediate Value Property

1. Let $[x]$ denote the integer part of the real number x. Suppose $f(x) = g(x)h(x)$ where $g(x) = [x^2]$ and $h(x) = \sin 2\pi x$. Discuss the continuity/discontinuity of f, g and h at $x = 2$ and $x = \sqrt{2}$.

2. Determine the points of continuity for the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational.} \end{cases}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $x_0, c \in \mathbb{R}$. Show that if $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

4. Let $f : [0, 1] \to (0, 1)$ be an on-to function. Show that f is not continuous on $[0, 1]$.

5. Let $f : [a, b] \to \mathbb{R}$ and for every $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| < \frac{1}{2}|f(x)|$. Find $\inf\{|f(x)| : x \in [a, b]\}$. Show that f is not continuous on $[a, b]$.

6. Let $f : [0, 2] \to \mathbb{R}$ be a continuous function and $f(0) = f(2)$. Prove that there exist real numbers $x_1, x_2 \in [0, 2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

7. Let p be an odd degree polynomial and $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$. Further show that the equation $x^{13} - 3x^{10} + 4x + sinx = \frac{1}{1+x^2} + cos^2x$ has a solution in $\mathbb{R}$.

Assignment 3 : Derivatives, Maxima and Minima, Rolle's Theorem

1. Show that the function $f(x) = x \mid x \mid$ is differentiable at 0. More generally, if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0.

2. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an even function (i.e., $f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative f' is an odd function (i.e.,$f(-x) = -f(x)$ for all $x \in \mathbb{R}$).

3. Show that among all triangles with given base and the corresponding vertex angle, the isosceles triangle has the maximum area.

4. Show that exactly two real values of x satisfy the equation $x^2 = xsinx + cosx$.

5. Suppose f is continuous on $[a,\ b]$, differentiable on $(a,\ b)$ and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. Then show that the equation $f'(x)f(x) = x$ has at least one root in $(a,\ b)$.

6. Let $f : (-1, 1) \to \mathbb{R}$ be twice differentiable. Suppose $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Show that $f'(0) = f''(0) = 0$.

7. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that $f''(0) > 0$. Show that there exists $n \in \mathbb{N}$ such that $f(\frac{1}{n}) \neq 1$.

Assignment 4 : Mean Value Theorem, Taylor's Theorem, Curve Sketching

1. Show that $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$ if $0 < y \leq x$, $n \in \mathbb{N}$.

2. Let $f : [0, 1] \to \mathbb{R}$ be differentiable, $f(\frac{1}{2}) = \frac{1}{2}$ and $0 < \alpha < 1$. Suppose $|f'(x)| \leq \alpha$ for all $x \in [0, 1]$. Show that $|f(x)| < 1$ for all $x \in [0, 1]$.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$ and $f(b) = b$. Show that there is $c \in (a, b)$ such that $f'(c) = 1$. Further, show that there are distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

4. Using Cauchy Mean Value Theorem, show that
   (a) $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.
   (b) $x - \frac{x^3}{3!} < \sin x$ for $x > 0$.

5. Find $\lim\limits_{x \longrightarrow 5} (6 - x)^{\frac{1}{x-5}}$ and $\lim\limits_{x \longrightarrow 0^+} (1 + \frac{1}{x})^x$.

6. Sketch the graphs of $f(x) = x^3 - 6x^2 + 9x + 1$ and $f(x) = \frac{x^2}{x^2-1}$.

7. (a) Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x_0 \in [a, b]$. Show that for any $x \in [a, b]$
   $$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$
   i.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.
   (b) Show that $\cos y - \cos x \geq (x - y) \sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

8. Suppose f is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem show that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.

Assignment 5 : Series, Power Series, Taylor Series

1. Let $f : [0, 1] \to \mathbb{R}$ and $a_n = f(\frac{1}{n}) - f(\frac{1}{n+1})$. Show that if f is continuous then $\sum_{n=1}^{\infty} a_n$ converges and if f is differentiable and $|f'(x)| < 1$ for all $x \in [0, 1]$ then $\sum_{n=1}^{\infty} |a_n|$ converges.

2. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$ where a_n equals:

(a) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$ (b) $1-\cos\frac{1}{n}$ (c) $2^{-n-(-1)^n}$ (d) $\left(1+\frac{1}{n}\right)^{n(n+1)}$

(e) $\frac{n\ln n}{2^n}$ (f) $\frac{logn}{n^p}, (p>1)$ (g) $e^{-n}(\cos n)n^2\sin\frac{1}{n}$

3. Let $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$ for all $n \ge N$. Show that if $\sum\limits_{n=1}^{\infty} b_n$ converges then $\sum\limits_{n=1}^{\infty} a_n$ also converges. Test the series $\sum\limits_{n=1}^{\infty} \frac{n^{n-2}}{e^n n!}$ for convergence.

4. Show that the series $\frac{1}{4^1}+\frac{1}{5^2}+\frac{3}{4^3}+\frac{1}{5^4}+\frac{5}{4^5}+\frac{1}{5^6}+\frac{7}{4^7}+\cdots$ converges.

5. Show that the series $\sum_{n=1}^{\infty}(-1)^n\sin\frac{1}{n}$ converges but not absolutely.

6. Determine the values of x for which the series $\sum\limits_{n=1}^{\infty}\frac{(x-1)^{2n}}{n^2 3^n}$ converges.

7. Show that $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},\ x\in\mathbb{R}$.

Assignment 6: Integration

1. Using Riemann's criterion for the integrability, show that $f(x)=\frac{1}{x}$ is integrable on $[1,2]$.

2. If f and g are continuous functions on $[a,\ b]$ and if $g(x)\ge 0$ for $a\le x\le b$, then show the mean value theorem for integrals : there exists $c\in[a,\ b]$ such that $\int\limits_a^b f(x)g(x)dx = f(c)\int\limits_a^b g(x)dx$.

   (a) Show that there is no continuous function f on $[0,1]$ such that $\int\limits_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}}$ for all $n\in\mathbb{N}$.

   (b) If f is contiunuous on $[a,b]$ then show that there exists $c\in[a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

   (c) If f and g are continuous on $[a,b]$ and $\int_a^b f(x)dx=\int_a^b g(x)dx$ then show that there exists $c\in[a,b]$ such that $f(c)=g(c)$.

3. Let $f:[0,2]\to\mathbb{R}$ be a continuous function such that $\int_0^2 f(x)dx = 2$. Find the value of $\int_0^2[xf(x)+\int_0^x f(t)dt]dx$.

4. Show that $\int\limits_0^x(\int\limits_0^u f(t)dt)du = \int\limits_0^x f(u)(x-u)du$, assuming f to be continuous.

5. Let $f:[0,1]\to\mathbb{R}$ be a positive continuous function. Show that $\lim_{n\to\infty}(f(\frac{1}{n})f(\frac{2}{n})\cdots f(\frac{n}{n}))^{\frac{1}{n}} = e^{\int_0^1 lnf(x)}$.

Assignment 7: Improper Integrals

1. Test the convergence/divergence of the following improper integrals:

   (a) $\int\limits_0^1 \frac{dx}{\log(1+\sqrt{x})}$ (b) $\int\limits_0^1\frac{dx}{x-\log(1+x)}$ (c) $\int\limits_0^1\frac{\log x}{\sqrt{x}}$ (d) $\int\limits_0^1\sin(1/x)dx$.

   (e) $\int\limits_1^{\infty}\frac{\sin(1/x)}{x}dx$ (f) $\int\limits_0^{\infty}e^{-x^2}dx$ (g) $\int\limits_0^{\infty}\sin x^2 dx,$ (h) $\int\limits_0^{\pi/2}\cot x dx$.

2. Determine all those values of p for which the improper integral $\int_0^\infty \frac{1-e^{-x}}{x^p} dx$ converges.

3. Show that the integrals $\int\limits_0^\infty \frac{\sin^2 x}{x^2} dx$ and $\int\limits_0^\infty \frac{\sin x}{x} dx$ converge. Further, prove that $\int\limits_0^\infty \frac{\sin^2 x}{x^2} dx = \int\limits_0^\infty \frac{\sin x}{x} dx$.

4. Show that $\int\limits_0^\infty \frac{x \log x}{(1+x^2)^2} dx = 0$.

5. Prove the following statements.

   (a) Let f be an increasing function on (0,1) and the improper integral $\int_0^1 f(x)$ exist. Then

      i. $\int_0^{1-\frac{1}{n}} f(x)dx \leq \frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-1}{n})}{n} \leq \int_{\frac{1}{n}}^1 f(x)dx$.

      ii. $\lim_{n\to\infty} \frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-1}{n})}{n} = \int_0^1 f(x)dx$.

   (b) $\lim_{n\to\infty} \frac{ln\frac{1}{n}+ln\frac{2}{n}+\cdots+ln\frac{n-1}{n}}{n} = -1$.

   (c) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.

Assignment 8: Applications of Integration, Pappus Theorem

1. Sketch the graphs of $r = \cos(2\theta)$ and $r = \sin(2\theta)$. Also, find their points of intersection.

2. A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

3. Let C denote the circular disc of radius b centered at $(a, 0)$ where $0 < b < a$. Find the volume of the torus that is generated by revolving C around the y-axis using

   (a) the Washer Method

   (b) the Shell Method.

4. Consider the curve C defined by $x(t) = \cos^3(t),\ y(t) = \sin^3 t,\ 0 \leq t \leq \frac{\pi}{2}$.

   (a) Find the length of the curve.

   (b) Find the area of the surface generated by revolving C about the x-axis.

   (c) If $(\overline{x}, \overline{y})$ is the centroid of C then find $\overline{y}$.

5. A square is rotated about an axis lying in the plane of the square, which intersects the square only at one of its vertices. For what position of the axis, is the volume of the resulting solid of revolution the largest?

6. Find the centroid of the semicircular arc $(x - r)^2 + y^2 = r^2,\ r > 0$ described in the first quadrant. If this arc is rotated about the line $y + mx = 0,\ m > 0$, determine the generated surface area A and show that A is maximum when $m = \pi/2$.

Assignment 9: Vectors, Curves, Surfaces, Vector Functions

1. Consider the planes $x - y + z = 1,\ x + ay - 2z + 10 = 0$ and $2x - 3y + z + b = 0$, where a and b are parameters. Determine the values of a and b such that the three planes

(a) intersect at a single point,

(b) intersect in a line,

(c) intersect (taken two at a time) in three distinct parallel lines.

2. Determine the equation of a cone with vertex $(0,-a,0)$ generated by a line passing through the curve $x^2 = 2y,\ z = h$.

3. The velocity of a particle moving in space is $\frac{d}{dt}c(t) = (\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}$. Find the particle's position as a function of t if $c(0) = 2\vec{i} + \vec{k}$. Also find the angle between its position vector and the velocity vector.

4. Show that $c(t) = \sin t^2\vec{i} + \cos t^2\vec{j} + 5\vec{k}$ has constant magnitude and is orthogonal to its derivative. Is the velocity vector of constant magnitude?

5. Find the point on the curve $c(t) = (5\sin t)\vec{i} + (5\cos t)\vec{j} + 12t\vec{k}$ at a distance 26π units along the curve from $(0,5,0)$ in the direction of increasing arc length.

6. Reparametrize the curves

   (a) $c(t) = \frac{t^2}{2}\vec{i} + \frac{t^3}{3}\vec{k},\ \ 0 \le t \le 2,$

   (b) $c(t) = 2\cos t\vec{i} + 2\sin t\vec{j},\ \ 0 \le t \le 2\pi$

   in terms of arc length.

7. Show that the parabola $y = ax^2,\ a \neq 0$ has its largest curvature at its vertex and has no minimum curvature.

Assignment 10: Functions of several variables (Continuity and Differentiability)

1. Identify the points, if any, where the following functions fail to be continuous:

$$(i)\ \ f(x,y) = \begin{cases} xy & \text{if } xy \ge 0 \\ -xy & \text{if } xy < 0 \end{cases} \qquad (ii)\ \ f(x,y) = \begin{cases} xy & \text{if } xy \text{ is rationnal} \\ -xy & \text{if } xy \text{ is irrational.} \end{cases}$$

2. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if}(x,y) = (0,0) \end{cases}$$

   Show that the function satisfy the following:

   (a) The iterated limits $\lim\limits_{x\longrightarrow 0}\left[\lim\limits_{y\longrightarrow 0} f(x,y)\right]$ and $\lim\limits_{y\longrightarrow 0}\left[\lim\limits_{x\longrightarrow 0} f(x,y)\right]$ exist and equals 0;

   (b) $\lim\limits_{(x,y)\longrightarrow(0,0)} f(x,y)$ does not exist;

   (c) $f(x,y)$ is not continuous at $(0,0)$;

   (d) the partial derivatives exist at $(0,0)$.

3. Let $f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0,0)$.

4. Let $f(x,y) = |xy|$ for all $(x,y) \in \mathbb{R}^2$. Show that

(a) f is differentiable at $(0, 0.)$

(b) $f_x(0, y_0)$ does not exist if $y_0 \neq 0$.

5. Suppose f is a function with $f_x(x, y) = f_y(x, y) = 0$ for all (x, y). Then show that $f(x, y) = c$, a constant.

Assignment 11: Directional derivatives, Maxima, Minima, Lagrange Multipliers

1. Let $f(x, y) = \frac{1}{2}\Big(\big|\,|x| - |y|\,\big| - |x| - |y|\Big)$. Is f continuous at $(0, 0)$? Which directional derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$?

2. Let $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that the directional derivative of f at $(0, 0)$ in all directions exist but f is not differentiable at $(0, 0)$.

3. Let $f(x, y) = x^2 e^y + \cos(xy)$. Find the directional derivative of f at $(1, 2)$ in the direction $(\frac{3}{5}, \frac{4}{5})$.

4. Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the z-axis.

5. Examine the following functions for local maxima, local minima and saddle points:

   $i)$ $4xy - x^4 - y^4$ $ii)$ $x^3 - 3xy^2$

6. Find the absolute maxima of $f(x, y) = xy$ on the unit disc $\{(x, y) : \ x^2 + y^2 \leq 1\}$.

Assignment 12 : Double Integrals

1. Evaluate the following integrals:

   $i)$ $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 - y^2}\, dydx$ $ii)$ $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dydx$ $iii)$ $\int_0^1 \int_y^1 x^2 \exp^{xy} dxdy$.

2. Evaluate $\iint_R x dxdy$ where R is the region $1 \leq x(1 - y) \leq 2$ and $1 \leq xy \leq 2$.

3. Using double integral, find the area enclosed by the curve $r = sin 3\theta$ given in polar cordinates.

4. Compute $\lim_{a \longrightarrow \infty} \int\int_{D(a)} \exp^{-(x^2+y^2)} dxdy$, where

   $i)$ $D(a) = \{(x, y) : x^2 + y^2 \leq a^2\}$ and $ii)$ $D(a) = \{(x, y) : 0 \leq x \leq a, \ 0 \leq y \leq a\}$.

   Hence prove that (i) $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (ii) $\int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$.

5. Find the volume of the solid which is common to the cylinder $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Assignment 13 : Triple Integrals, Surface Integrals, Line integrals

1. Evaluate the integral $\iiint_W \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}}$; where W is the ball $x^2 + y^2 + z^2 \leq 1$.

2. What is the integral of the function x^2z taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2+y^2=a^2$. What is the integral of the given function taken throughout the volume of the cylinder.

3. Find the line integral of the vector field $F(x,y,z) = y\vec{i} - x\vec{j} + \vec{k}$ along the path $\mathbf{c}(t) = (\cos t, \sin t, \frac{t}{2\pi}),\quad 0 \le t \le 2\pi$ joining $(1,0,0)$ to $(1,0,1)$.

4. Evaluate $\int\limits_C T \cdot dR$, where C is the circle $x^2+y^2=1$ and T is the unit tangent vector.

5. Show that the integral $\int\limits_C yzdx+(xz+1)dy+xydz$ is independent of the path C joining $(1,0,0)$ and $(2,1,4)$.

Assignment 14 : Green's /Stokes' /Gauss' Theorems

1. Use Green's Theorem to compute $\int\limits_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where C is the boundary of the region $\{(x,y) : x, y \ge 0 \;\&\; x^2+y^2 \le 1\}$.

2. Use Stokes' Theorem to evaluate the line integral $\int\limits_C -y^3\,dx + x^3\,dy - z^3\,dz$, where C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

3. Let $\overrightarrow{F} = \frac{\overrightarrow{r}}{|\overrightarrow{r}|^3}$ where $\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$ and let S be any surface that surrounds the origin. Prove that $\iint\limits_S \overrightarrow{F}.n\;d\sigma = 4\pi$.

4. Let D be the domain inside the cylinder $x^2+y^2=1$ cut off by the planes $z=0$ and $z=x+2$. If $\vec{F} = (x^2+ye^z, y^2+ze^x, z+xe^y)$, use the divergence theorem to evaluate $\iint\limits_{\partial D} F \cdot \mathbf{n}\;d\sigma$.