Lecture 29 : Mixed Derivative Theorem, MVT and Extended MVT

If $f : \mathbb{R}^2 \to \mathbb{R}$, then f_x is a function from $\mathbb{R}^2$ to $\mathbb{R}$ (if it exists). So one can analyze the existence of

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) \quad \text{and} \quad f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$$

which are partial derivatives of f_x with respect x or y and, similarly the existence of f_{yy} and f_{yx}. These are called second order partial derivatives of f.

The following example shows that, in general, f_{xy} need not be equal to f_{yx}.

Example : Let $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Note that

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{h(h^2-k^2)}{h^2+k^2} = h$$

and

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 1.$$

Similarly, $f_{xy}(0,0) = -1$.

Theorem 29.1(Mixed derivative theorem) : *If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined in a neighborhood of (x_0,y_0) and all are continuous at (x_0,y_0) then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.*

We will not present the proof of this result here. The proof is given in the text book.

Mean Value Theorem : We will present the MVT for functions of several variables which is a consequence of MVT for functions of one variable.

Theorem 29.2: *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Let $X_0 = (x_0,y_0)$ and $X = (x_0+h, y_0+k)$. Then there exists C which lies on the line joining X_0 and X such that*

$$f(X) = f(X_0) + f'(C)(X - X_0)$$

i.e, there exists $c \in (0,1)$ such that

$$f(x_0+h, y_0+k) = f(x_0,y_0) + hf_x(C) + kf_y(C) \text{ where } C = (x_0+ch, y_0+ck).$$

Proof : Define $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = f(x_0+th, y_0+tk), \ t \in [0,1].$$

Note that by the Chain Rule ϕ is differentiable and

$$\phi' = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} = hf_x + kf_y.$$

By the MVT, there exist $c \in (0,1)$ such that

$$\phi(1) - \phi(0) = \phi'(c).$$

This proves the result. □

Remark : In the previous result if we fix X_0 and X then it is enough to assume that the function f is differentiable on the line segment joining X and X_0.

Problem : *If $f(x,y)$ is constant if and only if $f_x = 0$ and $f_y = 0$.*

We will now take up the extended mean value theorem which we need.

Theorem 29.3(EMVT): *Let f, X, X_0 be as in the previous theorem. Suppose f_x and f_y are continuous and they have continuous partial derivatives. Then, there exists C which lies on the line joining X_0 and X such that*

$$f(X) = f(X_0) + f'(X_0)(X - X_0) + \frac{1}{2}(X - X_0)f''(C)(X - X_0)$$

where $f'' = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$. *That is, there exists $c \in (0,1)$ such that*

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + (hf_x + kf_y)(X_0) + \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})(C)$$

where $C = (x_0 + ch, y_0 + ck)$.

Proof (*): Consider the function $\phi(t)$ defined in the proof of the previous result. Since f_x and f_y are continuous f is differentiable. Therefore, as given in the proof of the previous theorem, ϕ is differentiable and

$$\phi' = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Since f_x and f_y have continuous partial derivatives, they are differentiable. Denote

$$\phi'(t) = hf_x(x_0 + th, y_0 + tk) + kf_y(x_0 + th, y_0 + tk) = F(x_0 + th, y_0 + tk),\ t \in [0,1].$$

Again by the Chain Rule,

$$\phi'' = hF_x + kF_y = h\frac{\partial}{\partial x}(hf_x + kf_y) + k\frac{\partial}{\partial y}(hf_x + kf_y) = h(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial y \partial x}) + k(h\frac{\partial^2 f}{\partial x \partial y} + k\frac{\partial^2 f}{\partial y^2}).$$

By the mixed derivative theorem,

$$\phi'' = h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}.$$

By the EMVT for ϕ, there exists $c \in (0,1)$ such that

$$\phi(1) = \phi(0) + \phi'(0) + \frac{\phi''(c)}{2}.$$

By substituting ϕ, ϕ' and ϕ'' in the above equation we get the result. □

Remarks : 1. We will consider f'', given in the statement of the previous theorem, as a notation. We do not say that the function f is twice differentiable.

2. We will recall the EMVT when we will deal with the second derivative test for local maxima and minima of $f(x,y)$ in the next lecture.

3. Whatever we discussed above can be generalized to the functions of three variables.

4. The matrix given in the statement of the previous theorem is called Hessian matrix. We should be able to guess what should be the corresponding Hessian matrix for the functions of three variables.

5. Note that we applied the MVT and the EMVT for the function ϕ to get the MVT and the EMVT for $f(x,y)$. Similarly by assuming that $f(x,y)$ has continuous partial derivatives of order n and applying Taylor's theorem for the function ϕ, we can obtain Taylor's Theorem for $f(x,y)$.