Lecture 14 : Power Series, Taylor Series

Let $a_n \in \mathbb{R}$ for $n = 0, 1, 2, \dots$. The series $\sum_{n=0}^{\infty} a_n x^n$, $x \in \mathbb{R}$, is called a power series. More generally, if $c \in \mathbb{R}$, then the series $\sum_{n=0}^{\infty} a_n (x-c)^n$, $x \in \mathbb{R}$, is called a power series around c. If we take $x' = x - c$ then the power series around c reduces to the power series around 0. In this lecture we discuss the convergence of power series.

Examples : 1. Consider the power series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Let us apply the ratio test and find the set of points in $\mathbb{R}$ on which the series converges. For any $x \in \mathbb{R}$, $\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|x|}{n+1} \to 0$ as $n \to \infty$. Therefore the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges absolutely for all $x \in \mathbb{R}$.

2. We know that the geometric series $\sum_{n=0}^{\infty} x^n$ converges only in $(-1, 1)$. Using the ratio test we can show that the series $\sum_{n=0}^{\infty} n! x^n$ converges only at $x = 0$.

The following result gives an idea about the set on which a power series converges.

Theorem 1: *Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for some x_0 and diverges for some x_1. Then*

(i) $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for all x such that $|x| < |x_0|$,*

(ii) $\sum_{n=0}^{\infty} a_n x^n$ *diverges for all x such that $|x| > |x_1|$.*

Proof (*): (i). Suppose $x_0 \neq 0$, $\sum_{n=0}^{\infty} a_n x_0^n$ converges and $|x| < |x_0|$. Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges, there exists $M \in \mathbb{R}$ such that $|a_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. Therefore,

$$| a_n x^n | = | a_n x_0^n || \frac{x}{x_0} |^n \leq M | \frac{x}{x_0} |^n \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $| \frac{x}{x_0} | < 1$, by comparison test, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

(ii) Let $x \in \mathbb{R}$ and $|x| > |x_1|$. Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges. Then by (i), the series $\sum_{n=0}^{\infty} a_n x_1^n$ converges absolutely which is a contradiction. □

From the above theorem, we can conclude that a power series $\sum_{n=0}^{\infty} a_n x^n$ is either converges for all $x \in \mathbb{R}$ or only at 0 or there is a unique $r, r > 0$ such that the series is absolutely converges for all x such that $|x| < r$ and diverges for all x such that $|x| > r$. This r is called the radius of convergence. In case the power series converges for all $x \in \mathbb{R}$ (resp., only at 0) then the radius of convergence of the series is ∞ (resp., 0).

If we define $S = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{is convergent}\}$, then the possibilities for S are

$$\{0\}, \qquad \mathbb{R}, \qquad (-r, r), \qquad [-r, r), \qquad (-r, r] \text{ and } [-r, r] \qquad \text{for some } r > 0.$$

Examples : 1. We have already seen above that the power series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all $x \in \mathbb{R}$ and hence the radius of convergence is ∞. Similarly the radius of convergence of $\sum_{n=0}^{\infty} n! x^n$ (resp., $\sum_{n=0}^{\infty} x^n$) is 0 (resp., 1).

2. Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n}$. Let us apply the ratio test to find the radius of convergence. For $x \in \mathbb{R}$ we have

$$\frac{| a_{n+1} x^{n+1} |}{| a_n x^n |} = | \frac{x^{n+1} n}{(n+1) x^n} | = | \frac{n}{n+1} x | \to | x | \quad \text{as} \quad n \to \infty.$$

It is clear that for $|x| < 1$ the series converges absolutely and diverges for $|x| > 1$. Therefore, the radius of convergence is 1 and the set $S = [-1, 1)$ which follows from the Leibniz test.

To find the radius of convergence of a power series or the set S, we use either the ratio test (as we did above) or root test. To find the sum of a convergent power series or for that matter sum of any convergent series is not easy. We will see sum of some particular type of power series called Taylor series.

Taylor Series : In one of the previous lectures we defined the nth degree Taylor polynomial $P_n(x)$ (w. r. to f and c), where

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

The power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n + ... \qquad \left(\text{or write} \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n\right)$$

is called the Taylor series of f around c. If $c = 0$, then the Taylor series of f around c is called Macluarin series.

If f is infinite times differentiable at c then the corresponding Taylor series is defined. Moreover, $P_n(x)$ is the nth partial sum of the Taylor series. We will see in the following examples that the Taylor series may not converge for all $x \in \mathbb{R}$ and even if it converges for some x, it need not converge to $f(x)$.

Examples : 1. If we consider the function $f : \mathbb{R}\backslash\{-1, 1\} \to \mathbb{R}$ given by $f(x) = 1/(1-x)$, then the Macluarin series is the geometric series $\sum_{n=0}^{\infty} x^n$ which converges on $(-1, 1)$.

2. Define $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Using L'Hospital rule, we can show that $f^{(k)}(0) = 0$ for all $k = 1, 2, ...$ Therefore the Macluarin series of f (for any $x \in \mathbb{R}$) is identically zero and it does not converge to $f(x)$ at any $x \neq 0$.

Taylor's theorem helps in showing the convergence of a Taylor series of f to $f(x)$ in the following way. Taylor's theorem says that there exists c_0 between x and c such that

$$E_n(x) \;=\; f(x) - P_n(x) \;=\; \frac{f^{(n+1)}(c_0)}{(n+1)!}(x-c)^{n+1}$$

It is clear from the above expression that if $E_n(x) \to 0$, then the Taylor series of f converges to $f(x)$ (as $P_n(x)$ is the nth partial sum of the Taylor series.)

Examples: Let $f(x) = sinx, x \in \mathbb{R}$. Then $\mid f^{(n)}(x) \mid \leq 1$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. In this case, the Macluarin series of f converges to $f(x)$ for all $x \in \mathbb{R}$ because $E_n(x) \to 0$. (One can use the ratio test for sequence to show that $E_n(x) \to 0$). So, we can expand the sin function in the series form on whole of $\mathbb{R}$ and we write $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R}$.

Similarly we can show that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \; x \in \mathbb{R}$.

Problem : *Show that* $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \; x > 0$.

Solution : Let $f(x) = e^x$. Fix $x > 0$. By Taylor,s Theorem there exists $c_n \in (0, x)$ such that

$$\mid E_n(x) \mid = \mid f(x) - (1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!}) \mid = \mid \frac{f^{n+1}(c_n)}{(n+1)!} x^{n+1} \mid = \frac{e^{c_n}}{(n+1)!} x^{n+1} \;\leq\; \frac{e^x}{(n+1)!} x^{n+1}.$$

Let $a_n = \frac{e^x}{(n+1)!} x^{n+1}$, then $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0$. This implies that $a_n \to 0$ and hence $E_n(x) \to 0$.