Lecture 19: Area between two curves; Polar coordinates

Recall that our motivation to introduce the concept of a Riemann integral was to define (or to give a meaning to) the area of the region under the graph of a function. If $f : [a, b] \to \mathbb{R}$ be a continuous function and $f(x) \geq 0$ then the area of the region between the graph of f and the x-axis is defined to be

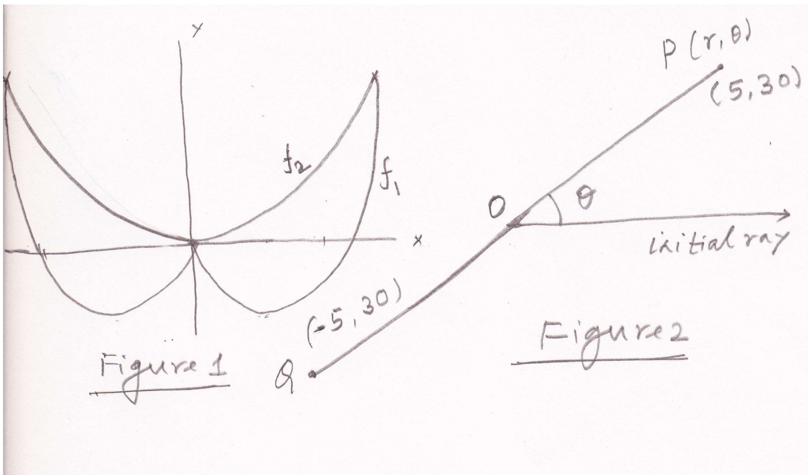
$$Area = \int_a^b f(x)dx.$$

Instead of the x-axis, we can take a graph of another continuous function $g(x)$ such that $g(x) \leq f(x)$ for all $x \in [a, b]$ and define the area of the region between the graphs to be

$$Area = \int_a^b (f(x) - g(x))\, dx.$$

Examples: 1. Let us find the area bounded by the curves: $f_1(x) = x^4 - 2x^2$ and $f_2(x) = 2x^2$. The common points of intersection of the graphs are the points satisfying : $f_1(x) = f_2(x)$ i.e., $x^4 - 2x^2 = 2x^2$, i.e., $x^4 - 4x^2 = 0$. Hence the points are $(0, 0), (2, 8), (-2, 8)$. It is understood that we have to find the area of the region given in Figure 1. The area is $\int_{-2}^{2}(f_2(x) - f_1(x))dx = \int_{-2}^{2}(2x^2 - x^4 + 2x^2)dx$.

2. Let us find the area bounded by the curves: $x = 3y - y^2$ and $x + y = 3$, i.e., $x = 3 - y$. The points of intersection are $(1, 1), (3, 0)$. Note that $(3y - y^2) - (3 - y) = -(y - 1)(y - 3) \geq 0$ for all $1 \leq y \leq 3$. Therefore the area is $\int_1^3 (3y - y^2) - (3 - y)dy$.



Polar Coordinates: To get a geometric idea we always relate a given function with a curve which is the graph of the given function. Sometimes we have to represent or express a given curve analytically (by a function or an equation). Expressing a given curve by the graph of a function or by an implicit equation using rectangular coordinates may not be always easy. Even if it is possible, in some cases, the function or the implicit equation may be complicated to use. Sometimes the polar coordinate system is better suited for the representation of a curve given geometrically. The term “curve” appearing here is the one which we usually imagine intuitively.

The polar coordinates are defined as follows. In the plane, we fix an origin O and an initial ray from O as shown in Figure 2. Then each point P in the plane can be assigned polar coordinates (r, θ) where r is the directed distance from O to P and θ is the directed angle from the initial ray to the segment OP.

The meaning of the directed angle is that the angle θ is positive when measured counterclockwise and negative when measured clockwise. The directed distance is something new. We will explain this concept with an example. Consider the points P and Q given in Figure 2. Here we assume that the lengths OP and OQ are same. Suppose $P = (5, 30)$, then Q is represented by $(-5, 30)$.

The negative distance can be understood as follows. If we go forward on the line QOP from O by the distance 5 we reach P and if we come backward on the same line from O by the distance 5 we reach Q. Note that Q has several representations, for example,

$$Q = (5, 210) = (-5, 30) = (5, -150) = (5, 570) = (-5, 390).$$

Of course one can ask what is the advantages of taking this directed distance (and the directed angle). We will take up the discussion on this question later.

Polar and Cartesian coordinates: If we use the common origin and take the initial ray as the positive x-axis, then the polar coordinates are related to the rectangular coordinates (x, y) by the equations:

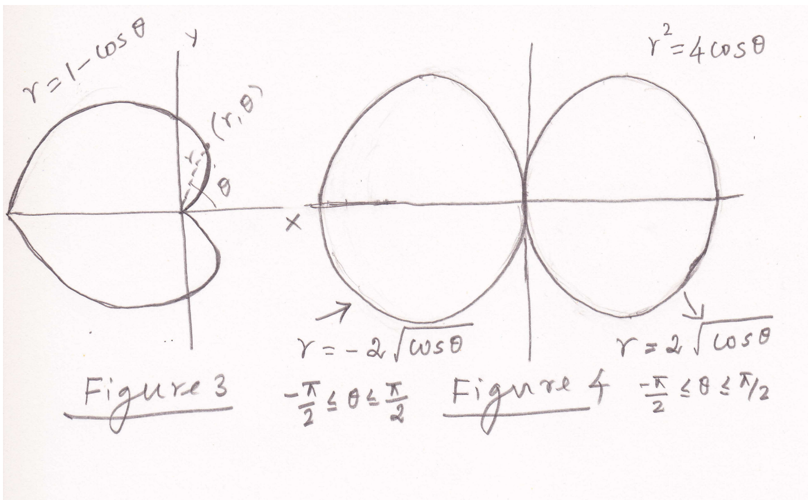
$$x = r\cos\theta,\ y = r\sin\theta \ \text{ or } \ x^2 + y^2 = r^2,\ \frac{y}{x} = \tan\theta.$$

Note that these equations are valid even if r is negative because $\cos(\theta+180) = -\cos\theta$, $\sin(\theta+180) = -\sin\theta$. The above equations are used to find Cartesian equations equivalent to polar equations and vice versa.

Example : $r^2 = 3r\sin\theta$ is equivalent to $x^2+y^2 = 3y$ which is a circle and $r\cos\theta = -4$ is equivalent to $x = -4$ which is a vertical line.

Graphs of the Polar Equations: A simple equation such as $r = 0$ (resp., $r = a$, $r = -a$, $\theta = \alpha$) represents the origin (resp., circle, the same circle, a straight line). We will now see how to represent the graph of a function given in polar equation: $r = f(\theta)$ or $F(r, \theta) = 0$. If the polar equation is given as $r = f(\theta)$, for sketching, we substitute a value of θ and find the corresponding $r = f(\theta)$. Then we plot the point (r, θ). To plot the curve we plot few points corresponding to few $\theta's$. To get the actual shape of the curve, it is desirable to consider the $\theta's$ for which $f(\theta)$ is a maximum or a minimum. As we do in the Cartesian case it is also desirable to consider the symmetry.

For example, the curve is symmetric about the origin (resp., x-axis, y-axis) if the equation is unchanged when r is replaced by $-r$ (resp., $-\theta$, $\pi-\theta$). The curve is also symmetric about the origin if the equation is unchanged when θ is replaced by $\theta + \pi$. Similarly, the curve is also symmetric about the x-axis (resp. y-axis) if the equation is unchanged when the pair (r, θ) is replaced by the pair $(-r, \pi - \theta)$ (resp., $(-r, -\theta)$).



Examples: 1. Let us sketch the curve $r = f(\theta) = a(1 - \cos\theta), a > 0$. Since $\cos(-\theta) = \cos\theta$. the curve is symmetric about the x-axis. We note that $0 \le r = f(\theta) \le 2a$ and $r = 0$ occur at $\theta = 0$ and $r = 2a$ occurs at $\theta = \pi$. Moreover, $1 - \cos\theta$ increases from 0 to 2. For $\theta = \pi/3$ and $\pi/2$, we have $r = a$ and $a/2$ respectively. With this information, we can plot the curve (see Figure 3).

2. Consider the equation $r^2 = 4\cos\theta$. This equation is not given in the form $r = f(\theta)$. The graph of this equation can be plotted in the following ways. By varying θ from 0 to $\pi/2$ we get the corresponding values of r. Since the curve is symmetric over the x-axis and y-axis, we get the curve as given in Figure 4. The other way is to convert the equation in the form $r = \pm 2\sqrt{\cos\theta}$ and sketch the graphs of the equations $r = +2\sqrt{\cos\theta}$ and $r = -2\sqrt{\cos\theta}$. We get one portion of the curve given in Figure 4 by plotting $(2\sqrt{\cos\theta}, \theta)$ for $-\pi/2 \le \theta \le \pi/2$ and the other by plotting $(-2\sqrt{\cos\theta}, \theta)$ for $-\pi/2 \le \theta \le \pi/2$. What would be the graph of the function $r^2 = -4\cos\theta$?

3. Consider the equation $r^2 = \sin 2\theta$. As we did in the previous example we can sketch the graph of $r = \pm\sqrt{sin 2\theta}$. Interestingly, in this case the graphs of $r = +\sqrt{\sin 2\theta}$ and $r = -\sqrt{\sin 2\theta}$ coincide. The graph is given in Figure 5.

4. Consider the equation $r = 3\sin\theta$. If we plot (r, θ) for $0 \le \theta \le \pi$, we get the curve given in Figure 6 and if we plot (r, θ) for $\pi \le \theta \le 2\pi$ we get the same curve.

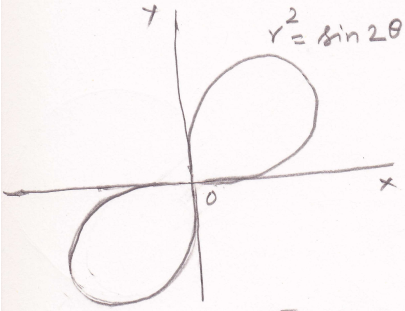


Figure 5

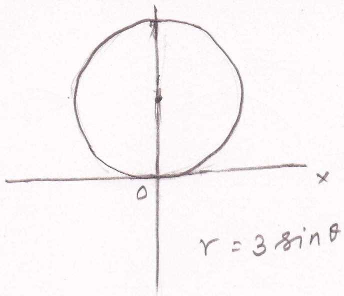


Figure 6

Remarks: 1. A point (r, θ) may not satisfy the equation $r = f(\theta)$ or $F(r, \theta) = 0$, however, it may still lie on the graph of the equation. For example $(2, \pi/2)$ does not satisfy the equation $r = 2\cos 2\theta$, however, $(2, \pi/2)$ lies on the curve, because $(-2, -\pi/2) = (2, \pi/2)$ and $(-2, -\pi/2)$ satisfies the equation. So the only sure way to identify all the points of intersection of two graphs is to sketch the graphs. Because solving of two equations may not lead to identifying all their points of intersection. We will see an example in the next lecture.

2. We will be dealing with the polar equations and their graphs only in the next one or two lectures. Later we will mainly use the polar coordinates to change the variables x and y to r and θ. In such cases we will assume $r > 0$ and $\theta \in [0, 2\pi)$, (at least we do not have to deal with the directed distance).

3. Allowing r to be negative has some advantages. For example, we could express the curve given in Figure 4 in a simple equation $r^2 = 4\cos\theta$. Several curves, especially those curves which are symmetric over the origin or the x-axis (see the lemniscate given in Figure 5), can be expressed in simpler forms if we allow the negative distance.

Note that the Cartesian equation $(x^2 + y^2)(x^2 + y^2)^2 = 16x^2$ is equivalent to the polar equation $r^2 = 4\cos\theta$. If we plot the points $(x, y)'s$ satisfying the Cartesian equation, we can see the symmetries over x-axis, y-axis and the origin. In fact the curves represented by the above Cartesian and the polar equations are same.