Lecture 5 : Existence of Maxima, Intermediate Value Property, Differentiabilty

Let f be defined on a subset S of $\mathbb{R}$. An element $x_0 \in S$ is called a *maximum* for f on S if $f(x_0) \geq f(x)$ for all $x \in S$ and in this case $f(x_0)$ is the maximum value f. Similarly, x_0 is called a *minimum* for f on S if $f(x_0) \leq f(x)$ for all $x \in S$.

The following theorem provides an existence criterion for maximum and minimum.

Theorem 5.1: *Let f be continuous on $[a,b]$. Then there exist $x_0, y_0 \in [a,b]$ such that x_0 is a maximum for f on $[a,b]$ and y_0 is a minimum for f on $[a,b]$.*

Proof (*): By Theorem 4.3, f is bounded on $[a,b]$. Let M be the least upper bound of $f([a,b])$ $(:= \{f(x) : x \in [a,b]\})$. Then there exists a sequence $\{f(x_n)\}$ in $f([a,b])$ such that $f(x_n) \to M$. Since $\{x_n\}$ is a bounded sequence in $[a,b]$, it has a convergent subsequence, say $x_{n_k} \to x_0 \in [a,b]$. By the continuity of f we have $f(x_0) = M$. Hence x_0 is a maximum for f on $[a,b]$.

The proof for the existence of a minimum is similar. □

Consider a function $f : [-1,1] \to \mathbb{R}$ such that f is continuous and satisfies $f(-1) < 0$ and $f(1) > 0$. Intuitively, we feel that the graph of f should cross the x-axis between -1 and 1; i.e., there is some $x_0 \in [-1,1]$ such that $f(x_0) = 0$. This motivates us to state the following theorem.

Theorem 5.2 (Intermediate Value Property) : *Let f be continuous on $[a,b]$, and let $f(a) < s < f(b)$ (s is a value which is intermediate between two values taken by f). Then there exists x such that $a < x < b$ and $f(x) = s$.*

Proof (*): Let $S = \{x \in [a,b] : f(x) \leq s\}$. Since $a \in S$, we have $S \neq \emptyset$ and S is bounded above by b. Let c be the least upper bound of S. We claim that $f(c) = s$. Since c is the least upper bound of S, there exists a sequence $\{x_n\}$ from S such that $x_n \to c$. By the continuity of f, $f(x_n) \to f(c)$. Since $f(x_n) \leq s$ for all n, we have $f(c) \leq s$. Note that $b > c$. Consider the sequence $y_n = c + (b-c)/n$. As $y_n \to c$, we get $f(y_n) \to f(c)$. Since $f(y_n) > s$ for all n, we have $f(c) \geq s$. It follows that $f(c) = s$. □

Problem 1 : *Show that the equation $(1-x)cosx = sinx$ has at least one solution in $(0,1)$.*

Solution: Set $f(x) = (1-x)cosx - sinx$. Then $f(0) = 1$ and $f(1) = -sin1 < 0$. By the intermediate value property there is $x_0 \in (0,1)$ such that $f(x_0) = 0$.

Problem 2 : *Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a fixed point in $[0,1]$; that is, there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$.*

Solution: Define the function $g(x) = f(x) - x$ on $[0,1]$. Then f is continuous, $g(0) \geq 0$ and $g(1) \leq 0$. Use the intermediate value property (IVP).

Problem 3 : *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Show that the range $\{f(x) : x \in [a,b]\}$ is a closed and bounded interval.*

Solution: Since f is a continuous function, there exist $x_0, y_0 \in [a,b]$ such that $f(x_0) = m = inff$ and $f(y_0) = M = supf$. Suppose $x_0 < y_0$. By the IVP, for every $\alpha \in [m,M]$ there exists $x \in [x_0, y_0]$ such that $f(x) = \alpha$. Hence $f([a,b]) = [m,M]$.

Problem 4 : *Show that a polynomial of odd degree has at least one real* root.

Solution: Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$ and n be odd. Then $p(x) =$

$x^n(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n})$. If $a_n > 0$, then $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to -\infty$ as $x \to -\infty$. Thus by the IVP, there exists x_0 such that $p(x_0) = 0$. Similar argument for $a_n < 0$.

Differentiation : We now deal with derivatives, an important concept of differential calculus. The reader must be familiar with this from elementary calculus. For example, the geometric problem of finding the tangent line to a curve at a given point leads to the notion of derivative.

Definition : Let I be an interval which is not a singleton and let f be a function defined on I. A function f is said to be differentiable at $x \in I$ if the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

It is understood that the limit is taken for $x + h \in I$; thus if x is a left end point of I then we only consider $h > 0$. If the above limit exists, it is called the derivative of f at x and is denoted by $f'(x)$. If f is differentiable at each $x \in I$, then f' is a function on I.

It is clear that if f is differentiable at $c \in I$, then

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

Now we prove that differentiability implies continuity.

Theorem 5.3 : *Let f be defined on an interval I. If f is differentiable at a point $c \in I$, then f is continuous at c.*

Proof: As $x \to c$, $x \neq c$, we have $f(x) - f(c) = \frac{f(x)-f(c)}{x-c}\ (x - c) \ \to\ f'(x) \cdot 0 = 0.$ □

Problem 5: *Show that the function $f(x)$ defined by $f(x) = x^2 sin\frac{1}{x}$, when $x \neq 0$ and $f(0) = 0$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not continuous at $x = 0$.*

Solution : Note that $f'(0) = \lim_{x\to 0} \frac{x^2 sin\frac{1}{x}}{x} = 0$. At other points the function is differentiable because the function is a product of two differentiable functions (here note that $sin\frac{1}{x}$ is a composition of two functions) and $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$, for $x \neq 0$. Since $\lim\limits_{h\to 0} \cos\frac{1}{h}$ does not exist, $f'(x)$ is not continuous at 0.

Problem 6: *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that $f'(c) = lim(n\{f(c + 1/n) - f(c)\})$. However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.*

Solution : Since $f'(c)$ exists, by taking $h = 1/n$ in the definition of the differentiability we get that $f'(c) = lim(n\{f(c + 1/n) - f(c)\})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for x rational, $f(x) = 1$ for x irrational. Then the function is not even continuous at 0; hence it cannot be differentiable at 0. However, $lim(n\{f(c + 1/n) - f(c)\})$ exists at $c = 0$.

Problem 7: *Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer k, show that*

$$\lim_{x\to 0} \frac{1}{x}\left\{f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + ... + f(\frac{x}{k})\right\} = 1 + \frac{1}{2} + ... + \frac{1}{k}.$$

Solution : $\lim\limits_{x\to 0} \frac{1}{x}(f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + \cdots + f(\frac{x}{k})) =$

$$\lim_{x\to 0}(\frac{f(x)-f(0)}{x} + \frac{1}{2}\frac{f(\frac{x}{2})-f(0)}{\frac{x}{2}} + \cdots + \frac{1}{k}\frac{f(\frac{x}{k})-f(0)}{\frac{x}{k}}) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$