Practice Problems 10 : Taylor's Theorem

1. Let $f : [a, b] \to \mathbb{R}$ and n be a non-negative integer. Suppose that $f^{(n+1)}$ exists on $[a, b]$. Show that f is a polynomial of degree $\leq$ n if $f^{(n+1)}(x) = 0$ for all $x \in [a, b]$. Observe that the statement for $n = 0$ can be proved by the mean value theorem.

2. Show that $1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$ for $x > 0$.

3. Show that for $x > 0$, $\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) \right| \leq \frac{x^4}{4}$.

4. Show that for $x \in \mathbb{R}$ with $|x|^5 < \frac{5!}{10^4}$, we can replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude less than or equal to 10^{-4}.

5. Prove the binomial expansion: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + x^n, \; x \in \mathbb{R}$

6. Using Taylor's theorem compute $\lim_{x \to 0} \frac{1 - \sqrt{1+x^2}\cos x}{x^4}$.

7. (a) Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x_0 \in [a, b]$. Show that for any $x \in [a, b]$
$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$
i.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.
   (b) Show that $\cos y - \cos x \geq (x - y)\sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

8. (a) Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose that $x, y \in (a, b)$, $x < y$ and $0 < \lambda < 1$. Show that
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$
i.e., the chord joining the two points $(x, f(x))$ and $(y, f(y))$ lies above the portion of the graph $\{(z, f(z)) : z \in (x, y)\}$.
   (b) Show that $\lambda \sin x \leq \sin \lambda x$ for all $x \in [0, \pi]$ and $0 < \lambda < 1$.

9. Let $f : [a, b] \to \mathbb{R}$ be twice differentiable. Suppose $f'(a) = f'(b) = 0$. Show that there exist $c_1, c_2 \in (a, b)$ such that $|f(b) - f(a)| = \left(\frac{b-a}{2}\right)^2 \frac{1}{2} \, |f''(c_1) - f''(c_2)|$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'''(x) > 0$ for all $x \in \mathbb{R}$. Suppose that $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. Show that $f(x_2) - f(x_1) > f'\left(\frac{x_1+x_2}{2}\right)(x_2 - x_1)$.

11. Let f be a twice differentiable function on $\mathbb{R}$ such that $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Show that if f is bounded then it is a constant function.

12. (a) For a positive integer n, show that there exists $c \in (0, 1)$ such that
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{e^c}{(n+1)!}.$$
Further show that $\frac{e^c}{n+1} = n!e - m$ for some integer m.
   (b) **(*)** Show that e is an irrational number.

13. **(*)** Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Suppose f is strictly increasing and $f(\overline{x}) = 0$. Let $x_0 > \overline{x}$ and (x_n) be the sequence generated by Newton's algorithm with the initial point x_0.
   (a) Show that $f'(x_0) > 0$.
   (b) Show that (x_n) converges.

Practice Problems 10: Hints/Solutions

1. Take any $x \in (a, b]$ and apply Taylor's Theorem for f on $[a, x]$. We get that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$ which is a polynomial of degree $\leq n$.

2. By Taylor's theorem there exists $c \in (0, x)$ such that $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}\frac{x^2}{(1+c)^{3/2}}$.

3. By Taylor's theorem there exists $c \in (0, x)$ such that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4(1+c)^4}$.

4. There exists $c \in (0, x)$ such that $\sin x = x - \frac{x^3}{3!} + (\cos c)\frac{x^5}{5!}$. If $|x|^5 < \frac{5!}{10^4}$, then $|\sin x - \left(x - \frac{x^3}{6}\right)| \leq 10^{-4}$.

5. Apply Taylor's theorem for $f(x) = x^n$ on $[1, 1+x]$ when $x > 0$ and $[1+x, 1]$ when $x < 0$.

6. Observe from Taylor's theorem that $\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \alpha x^6$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \beta x^5$ for some α and β in $\mathbb{R}$. The limit is $\frac{1}{3}$.

7. (a) There exists c between x_0 and x such that $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{(x-x_0)^2}{2}f''(c)$. This implies the required inequality.

   (b) Take $f(x) = \cos x$ on $[\frac{\pi}{2}, \frac{3\pi}{2}]$ and apply the inequality given in (a).

8. (a) Let $x_\lambda = \lambda x + (1-\lambda)y$. By Problem 7(a), $f(x) \geq f(x_\lambda) + f'(x_\lambda)(1-\lambda)(x-y)$ and $f(y) \geq f(x_\lambda) + f'(x_\lambda)\lambda(y-x)$. Eliminate $f'(x_\lambda)$.

   (b) Take $f(x) = -\sin x$ on $[0, \pi]$ and apply the inequality given in (a).

9. By Taylor's theorem $f\left(\frac{a+b}{2}\right) = f(a) + \frac{f''(c_1)}{2}\left(\frac{b-a}{2}\right)^2$ and $f\left(\frac{a+b}{2}\right) = f(b) + \frac{f''(c_2)}{2}\left(\frac{b-a}{2}\right)^2$ for some $c_1, c_2 \in (a, b)$. Eliminate $f\left(\frac{a+b}{2}\right)$.

10. Let $\overline{x} = \frac{x_1+x_2}{2}$. Since $f'''(x) > 0$ for all $x \in \mathbb{R}$, by Taylor's theorem $f(x_2) > f(\overline{x}) + f'(\overline{x})(x_2-\overline{x}) + \frac{f''(\overline{x})}{2}(x_2-\overline{x})^2$ and $f(x_1) < f(\overline{x}) + f'(\overline{x})(x_1-\overline{x}) + \frac{f''(\overline{x})}{2}(x_2-\overline{x})^2$. Eliminate $f(\overline{x})$ and $\frac{f''(\overline{x})}{2}(x_2-\overline{x})^2$.

11. Suppose $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Since $f''(x) \geq 0$ for all $x \in \mathbb{R}$, by Problem 7(a), $f(x) \geq f(x_0) + f'(x_0)(x-x_0) \to \infty$ as $x \to \infty$. This contradicts the fact that f is bounded.

12. (a) For $f(x) = e^x$ on $[0, 1]$, by Taylor's theorem, there exists $c \in (0, 1)$ such that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{e^c}{(n+1)!}$. Multiply both sides by $n!$ to get $\frac{e^c}{n+1} = n!e - m$ for some integer m.

    (b) If $e = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then by (a), $\left(\frac{p}{q}\right)^c \frac{1}{n+1} = n!\frac{p}{q} - m$. Since $n!\frac{p}{q} - m$ is an integer for $n \geq q$, $\left(\frac{p}{q}\right)^c \frac{1}{n+1}$ is a natural number for every $n \geq q$. But $\left(\frac{p}{q}\right)^c \frac{1}{n+1} \to 0$ as $n \to \infty$ which is a contradiction.

13. (a) Observe that $f'(x) \geq 0 \; \forall \; x \in \mathbb{R}$ because f is strictly increasing. Note that f' is also increasing. If $f'(x_0) = 0$, then $f'(x) = 0 \; \forall \; x \leq x_0$. That is f is constant on $(-\infty, x_0]$ which is not true.

    (b) Since $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, by (a), $x_1 \leq x_0$. By Problem 7(a), $f(x_1) - f(x_0) \geq f'(x_0)(x_1 - x_0) = -f(x_0)$ and hence $f(x_1) \geq 0$. Therefore $\overline{x} \leq x_1 \leq x_0$. Similarly we can show that $\overline{x} \leq ... \leq x_n \leq ... \leq x_2 \leq x_1 \leq x_0$. Therefore the sequence (x_n) is decreasing and bounded below.