

Practice Problems 11 : Series: Definition of convergence, Necessary and sufficient conditions
for convergence, Absolute convergence.

1. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$ converges for any $p \in \mathbb{N}$.
2. Show that $\sum_{n=1}^{\infty} (a_n + b_n)$ converges if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.
3. Show that every sequence is a sequence of partial sums of a series.
4. Show that $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ converges if and only if the sequence (a_n) converges. Verify the convergence/divergence of the following series:
 - (a) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$
 - (b) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$
5. Show that the series $\frac{1}{2} + \frac{1}{3} + \frac{2}{3} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{1}{6} + \frac{2}{6} + \dots$ diverges.
6. Let (a_n) be a sequence such that $a_n > 0$ and $a_n \leq a_{2n} + a_{2n+1}$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} a_n$ diverges.
7. Consider the sequence 0.2, 0.22, 0.222, 0.2222. By writing this sequence as a sequence of partial sums of a series, find the limit of this sequence.
8. Let $\sum_{n=1}^{\infty} a_n$ converge and $a_n > 0$ for all n . If (a_{n_k}) is a subsequence of (a_n) , show that $\sum_{k=1}^{\infty} a_{n_k}$ also converges.
9. Show that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\sum_{i=m}^n a_i| < \epsilon$ for all $m, n \in \mathbb{N}$ satisfying $n \geq m \geq N$.
10. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Consider $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ where $b_n = \max\{a_n, 0\}$ and $c_n = \min\{a_n, 0\}$ (i.e., series of positive terms and series of negative terms of $\sum_{n=1}^{\infty} a_n$).
 - (a) If $\sum_{n=1}^{\infty} |a_n|$ converges then show that both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ converge.
 - (b) If $\sum_{n=1}^{\infty} |a_n|$ diverges then show that both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ diverge.
11. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and $\sum_{n=1}^{\infty} b_n$ is obtained by grouping finite number of terms of $\sum_{n=1}^{\infty} a_n$ such as $(a_1 + a_2 + \dots + a_{m_1}) + (a_{m_1+1} + a_{m_1+2} + \dots + a_{m_2}) + \dots$ for some m_1, m_2, \dots (Here $b_1 = a_1 + a_2 + \dots + a_{m_1}$, $b_2 = a_{m_1+1} + a_{m_1+2} + \dots + a_{m_2}$ and so on). Show that $\sum_{n=1}^{\infty} b_n$ converges and has the same limit as $\sum_{n=1}^{\infty} a_n$. What happens if $\sum_{n=1}^{\infty} a_n$ diverges ?
12. Let $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ be a convergent series. Suppose $\sum_{n=1}^{\infty} b_n$ is obtained by rearranging the terms of $\sum_{n=1}^{\infty} a_n$ (i.e., the terms of $\sum_{n=1}^{\infty} b_n$ are same as those of $\sum_{n=1}^{\infty} a_n$ but they occur in different order). Show that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to the same limit.
13. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(-)^{n+1}}{n}$. Show that the series
$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \dots,$$
which is obtained from $\sum_{n=1}^{\infty} a_n$ by rearranging and grouping, is $\frac{1}{2} \sum_{n=1}^{\infty} a_n$.
14. (*) Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(-)^{n+1}}{n}$. Let $\alpha \in \mathbb{R}$; for example take $\alpha = 2013$.

(a) Show that there exists a smallest odd positive integer N_1 such that $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} > 2013$. Further show that $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} \leq 2013$.

(b) Show that there exists a smallest odd positive integer $N_2 > N_1$ such that

$$1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2} > 2013.$$

Further show that $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2} - \frac{1}{4} \leq 2013$.

(c) Show that $0 \leq \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2}\right) - 2013 \leq \frac{1}{N_2}$ and $0 \leq 2013 - \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2} - \frac{1}{4}\right) \leq \frac{1}{4}$.

(d) Following (b), consider the series of rearrangement

$$1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2} - \frac{1}{4} + \frac{1}{N_2+2} + \dots + \frac{1}{N_3} - \frac{1}{6} + \dots$$

Show that

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2} - \frac{1}{4} + \frac{1}{N_2+2} + \dots + \frac{1}{N_3}\right) - 2013 \leq \frac{1}{N_3}.$$

Further, for any j such that $N_2 + 2 \leq N_2 + 2j \leq N_3 - 2$, show that

$$2013 - \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2} - \frac{1}{4} + \frac{1}{N_2+2} + \dots + \frac{1}{N_2+2j}\right) \leq \frac{1}{4}.$$

(e) Show that the series of rearrangement given in (d) converges to 2013.

15. (*) Let (A_n) and (S_n) be the sequences of partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{n}$ respectively. If $\sum_{n=1}^{\infty} a_n$ is convergent or the sequence of partial sums (A_n) is bounded then show that

(a) $S_n = A_1\left(\frac{1}{1} - \frac{1}{2}\right) + A_2\left(\frac{1}{2} - \frac{1}{3}\right) + \dots + A_n\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{A_n}{n+1}$, for $n > 1$,

(b) the series $|A_1\left(\frac{1}{1} - \frac{1}{2}\right)| + |A_2\left(\frac{1}{2} - \frac{1}{3}\right)| + \dots + |A_n\left(\frac{1}{n} - \frac{1}{n+1}\right)| + \dots$ converges,

(c) the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Practice Problems 11 : Hints/Solutions

1. Let (S_n) be the sequence of partial sums of $\sum_{n=p}^{\infty} a_n$. Then for all $n > p$, the n th term of the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$ is $a_1 + a_2 + \dots + a_{p-1} + S_n$. Use the definition of the convergence of a series.
2. Use the definition of the convergence of a series.
3. Let (a_n) be the given sequence. Consider the series $a_1 + (a_2 - a_1) + (a_3 - a_2) + \dots$
4. Note that the sequence of partial sums of the series $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ is $(a_1 - a_n)$.
5. The n th term of the series does not converge to 0.
6. Let (S_n) be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$. Consider, for example, $S_7 = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) \geq a_1 + a_1 + (a_2 + a_3) \geq a_1 + a_1 + a_1$. Show that (S_n) is unbounded.

7. The sequence is $\frac{2}{10}, \frac{2}{10} + \frac{2}{10^2}, \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3}, \dots$ which is a sequence of partial sums of the series $\sum_{n=1}^{\infty} \frac{2}{10^n}$. The given sequence converges to $\frac{2}{9}$.
8. The sequence of partial sums of $\sum_{k=1}^{\infty} a_{n_k}$ increases and bounded above.
9. Use the fact that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums (S_n) satisfies the Cauchy criterion.
10. (a) Observe that $2b_n = a_n + |a_n|$ and $2c_n = a_n - |a_n|$ for all $n \in \mathbb{N}$.
 (b) Observe that $|a_n| = 2b_n - a_n$ and $|a_n| = a_n - 2c_n$ for all $n \in \mathbb{N}$.
11. Let (S_n) and $(\overline{S_k})$ be the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Observe that $(\overline{S_k})$ is a subsequence of (S_n) . For the next part, consider the series $1 - 1 + 1 - 1 + 1 - 1 \dots$ and the grouping $(1 - 1) + (1 - 1) + (1 - 1) + \dots$
12. Let (S_n) and $(\overline{S_n})$ be the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Note that both (S_n) and $(\overline{S_n})$ are increasing sequences. Suppose $S_n \rightarrow S$ for some S . Then $\overline{S_n} \leq S$ for all n . Therefore $\overline{S_n}$ converges. If $\overline{S_n} \rightarrow \overline{S}$ for some \overline{S} , then $\overline{S} \leq S$. For the proof of $S \leq \overline{S}$, interchange a_n and b_n .
13. Trivial.
14. (a) Since the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ diverges, the sequence of the partial sums is unbounded. Therefore there exists a smallest odd positive integer N_1 such that $1 + \frac{1}{3} + \dots + \frac{1}{N_1} > 2013$. If $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1-2} + \frac{1}{N_1} - \frac{1}{2} > 2013$, then $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1-2} > 2013$ as $\frac{1}{N_1} - \frac{1}{2} < 0$ which is a contradiction.
 (b) Similar to (a).
 (c) From (b), it follows that $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2-2} \leq 2013$. That is $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2-2} + \frac{1}{N_2} \leq 2013 + \frac{1}{N_2}$. This implies the first inequality of (c).
 From the first inequality of (b), $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \dots + \frac{1}{N_2} - \frac{1}{4} > 2013 - \frac{1}{4}$. This implies the second inequality of (c).
 (d) The proof of the first part is similar to the proof of the first part of (c). The second part follows from the second part of (c).
 (e) Observe from (c) and (d) that the sequence of partial sums of the series of rearrangement converges to 2013.
15. (a) Use the fact that $a_n = A_n - A_{n-1}$.
 (b) Since (A_n) is a bounded sequence, let $|A_n| \leq M$ for all $n \in \mathbb{N}$ and for some M . Therefore $|A_1(\frac{1}{1} - \frac{1}{2})| + |A_2(\frac{1}{2} - \frac{1}{3})| + \dots + |A_n(\frac{1}{n} - \frac{1}{n+1})| \leq M(1 - \frac{1}{n+1}) < M$.
 (c) From (b), the sequence of partial sums of the series $A_1(\frac{1}{1} - \frac{1}{2}) + A_2(\frac{1}{2} - \frac{1}{3}) + \dots$ converges. Therefore (S_n) converges.