Practice Problems 2: Convergence of sequences and monotone sequences

1. Investigate the convergence of the sequence (x_n) where

   (a) $x_n = \frac{1}{1+n^2} + \frac{2}{2+n^2} + ... + \frac{n}{n+n^2}$.

   (b) $x_n = (a^n + b^n)^{1/n}$ where $0 < a < b$.

   (c) $x_n = (\sqrt{2} - 2^{\frac{1}{3}})(\sqrt{2} - 2^{\frac{1}{5}})...(\sqrt{2} - 2^{\frac{1}{2n+1}})$.

   (d) $x_n = n^\alpha - (n+1)^\alpha$ for some $\alpha \in (0, 1)$.

   (e) $x_n = \frac{2^n}{n!}$.

   (f) $x_n = \frac{1-2+3-4+\cdots+(-1)^{n-1}n}{n}$.

2. Let $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) does not converge.

3. Let A be a non-empty subset of $\mathbb{R}$ and $\alpha = \inf A$. Show that there exists a sequence (a_n) such that $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \alpha$.

4. Let $x_0 \in \mathbb{Q}$. Show that there exists a sequence (x_n) of irrational numbers such that $x_n \to x_0$.

5. Let (x_n) be a sequence in $\mathbb{R}$. Prove or disprove the following statements.

   (a) If $x_n \to 0$ and (y_n) is a bounded sequence then $x_n y_n \to 0$.

   (b) If $x_n \to \infty$ and (y_n) is a bounded sequence then $x_n y_n \to \infty$.

6. Let (x_n) be a sequence in $\mathbb{R}$. Prove or disprove the following statements.

   (a) If the sequence $(x_n + \frac{1}{n} x_n)$ converges then (x_n) converges.

   (b) If the sequence $(x_n^2 + \frac{1}{n} x_n)$ converges then (x_n) converges.

7. Show that the sequence (x_n) is bounded and monotone, and find its limit where

   (a) $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$.

   (b) $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for $n \in \mathbb{N}$.

   (c) $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$, for $n \in \mathbb{N}$.

8. Let $0 < b_1 < a_1$ and define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$ for all $n \in \mathbb{N}$. Show that both (a_n) and (b_n) converge.

9. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $\sqrt{a}$.

10. Let (x_n) be a sequence in $(0, 1)$. Suppose $4x_n(1 - x_{n+1}) > 1$ for all $n \in \mathbb{N}$. Show that the sequence is monotone and find the limit.

11. Let A be a non-empty subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that there exists a sequence (a_n) in A such that $|x_0 - a_n| \to d(x_0, A)$. Recall that $d(x_0, A) = inf\{|x_0 - a| : a \in A\}$.

12. Let (a_n) be a bounded sequence. For every $n \in \mathbb{N}$, define $x_n = \sup\{a_k : k > n\}$. Show that the sequence (x_n) converges.

13. **(*)** Show that the sequence (e_n) defined by $e_n = (1 + \frac{1}{n})^n$ is increasing and bounded above.

Practice Problems2: Hints/Solutions

1. (a) Since $(1+2+...+n)\frac{1}{n+n^2} \leq x_n \leq (1+2+...+n)\frac{1}{1+n^2}$, $x_n \to \frac{1}{2}$

   (b) Note that $b = (b^n)^{1/n} \leq x_n \leq (2b^n)^{1/n} = 2^{1/n}b \to b$. Therefore $x_n \to b$.

   (c) We have $0 < x_n < (\sqrt{2}-1)^n$ and hence $x_n \to 0$.

   (d) Observe that $-x_n = n^\alpha[(1+\frac{1}{n})^\alpha - 1] < n^\alpha[1+\frac{1}{n}-1] = \frac{1}{n^{1-\alpha}} \to 0$. Hence $x_n \to 0$.

   (e) Consider $\frac{x_{n+1}}{x_n}$ and apply the ratio test for sequences to conclude that $x_n \to 0$.

   (f) Here $x_{2n} = -\frac{1}{2}$ and $x_{2n+1} = \frac{n+1}{2n+1} \to \frac{1}{2}$. The sequence does not converge.

2. Suppose $x_n \to x_0$ for some x_0. Then, by the definition, for $\epsilon = \frac{1}{4}$ (why $\frac{1}{4}$?) there exists N such that $|x_n - x_0| < \frac{1}{4}$ for all $n \geq N$. Then for all $m, n \geq N$, $|x_n - x_m| \leq |x_n - x_0| + |x_m - x_0| \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ which is not true because $|x_n - x_{n+1}| = 2$ for any n.

3. Since $\alpha + \frac{1}{n}$ is not a l.b., find $a_n \in A$ such that $\alpha \leq a_n < \alpha + \frac{1}{n}$. Allow $n \to \infty$.

4. Find an irrational x_n satisfying $x_0 < x_n < x_0 + \frac{1}{n}$ for every $n \in \mathbb{N}$. Allow $n \to \infty$.

5. (a) True. Find $M \in \mathbb{N}$ such that $0 \leq |x_n y_n| < M|x_n|$. Allow $n \to \infty$.

   (b) False. Take $x_n = n$ and $y_n = \frac{1}{n}$.

6. (a) Let $y_n = x_n + \frac{1}{n}x_n = (1+\frac{1}{n})x_n$. Then $x_n = \frac{y_n}{(1+\frac{1}{n})}$. Hence (x_n) converges if (y_n) converges.

   (b) The statement is not true. Take, for example, $x_n = (-1)^n$.

7. (a) Observe that $x_2 < x_1$. If $x_n < x_{n-1}$, then $x_{n+1} < 2 - \frac{1}{x_{n-1}} = x_n$. The sequence is decreasing. Note that $x_n > 0$. The sequence converges and the limit is 1.

   (b) Observe that $x_2 > x_1$. Since $x_{n+1}^2 - x_n^2 = 2(x_n - x_{n-1})$, by induction (x_n) is increasing. It can be observed again by induction that $x_n \leq 2$. The limit is 2.

   (c) Note that $x_2 > x_1$. Since $x_{n+1} - x_n = \frac{x_n - x_{n-1}}{(3+2x_n)(3+2x_{n-1})}$, by induction (x_n) is increasing. Note that $x_{n+1} = 1 + \frac{1+x_n}{3+2x_n} \leq 2$. The limit is $\sqrt{2}$.

8. By the AM-GM inequality $b_n \leq a_n$. Therefore $0 \leq a_{n+1} \leq \frac{a_n + a_n}{2} = a_n$. Note that $b_{n+1} \geq \sqrt{b_n b_n} = b_n$ and $b_n \leq a_n \leq a_1$. Use monotone criterion for both (a_n) and (b_n).

9. Note that $x_n > 0$ and $x_{n+1} - x_n = \frac{1}{2}(x_n + \frac{a}{x_n}) - x_n = \frac{1}{2}(\frac{a - x_n^2}{x_n})$. Further, by the A.M -G.M. inequality, $x_{n+1} \geq \sqrt{a}$. Therefore (x_n) is decreasing and bounded below.

10. By the AM-GM inequality $\frac{x_n + (1 - x_{n+1})}{2} \geq \sqrt{x_n(1 - x_{n+1})} > \frac{1}{2}$. Therefore $x_n > x_{n+1}$. Suppose $x_n \to x_0$ for some x_0. Then $4x_0(1 - x_0) \geq 1$ which implies that $(2x_0 - 1)^2 \leq 0$. Therefore $x_0 = \frac{1}{2}$.

11. Use Problem 2 or follow the steps of the solution of Problem 2.

12. Observe that the sequence (x_n) is decreasing and bounded.

13. By binomial theorem $e_n = 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + ... + \frac{1}{n!}(1-\frac{1}{n})...(1-\frac{n-1}{n}) \leq e_{n+1}$ and $e_n \leq 2 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} \leq 2 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} \leq 3$.

    Alternate Solution: For each $n \in \mathbb{N}$, apply AM-GM inequality for $a_1 = 1, a_2 = a_3 = ... = a_{n+1} = 1 + \frac{1}{n}$. We get $e_{n+1} > e_n$.