PP 34 : Triple integral, Change of variables, Cylindrical and Spherical coordinates

1. Let D denote the solid bounded by the surfaces $y = x$, $y = x^2$, $z = x$ and $z = 0$. Evaluate $\iiint\limits_D y\,dxdydz$.

2. Let D denote the solid bounded below by the plane $z + y = 2$, above by the cylinder $z + y^2 = 4$ and on the sides $x = 0$ and $x = 2$. Evaluate $\iiint\limits_D x\,dxdydz$.

3. Suppose $\int\limits_0^4 \int\limits_{\sqrt{x}}^2 \int\limits_0^{2-y} dzdydx = \iiint\limits_D dxdydz$ for some region $D \subset \mathbb{R}^3$.

   (a) Sketch the region D.

   (b) Sketch the projections of D on the xy, yz and xz planes.

   (c) Write $\int\limits_0^4 \int\limits_{\sqrt{x}}^2 \int\limits_0^{2-y} dzdydx$ as iterated integrals of other orders.

4. Let $D = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1\}$ and $E = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leq 1\}$. Show that $\iiint_D dxdydz = \iiint_E 24dudvdw$.

5. In each of the following cases, describe the solid D in terms of the cylindrical coordinates.

   (a) Let D be the solid that is bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

   (b) Let D be the solid that lies within the cylinder $x^2 + (y-1)^2 = 1$ below the paraboloid $z = x^2 + y^2$ and above the plane $z = 0$.

   (c) Let S denote the torus generated by revolving the circle $\{(x, z) : (x - 2)^2 + z^2 = 1\}$ about the z-axis. Let D be the solid that is bounded above by the surface S and below by $z = 0$.

6. Let D be the solid that lies inside the cylinder $x^2+y^2 = 1$, below the cone $z = \sqrt{4(x^2 + y^2)}$ and above the plane $z = 0$. Evaluate $\iiint\limits_D x^2dxdydz$.

7. Evaluate $\int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int\limits_{x^2+y^2}^{4} xdzdydx$.

8. Describe the following regions in terms of the spherical coordinates.

   (a) The region that lies inside the sphere $x^2 + y^2 + (z - 2)^2 = 4$ and outside the sphere $x^2 + y^2 + z^2 = 1$.

   (b) The region that lies below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z = \sqrt{x^2 + y^2}$.

   (c) The region that is enclosed by the cone $z = \sqrt{3(x^2 + y^2)}$ and the planes $z = 1$ and $z = 2$.

9. Let D denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$. Evaluate $\iiint\limits_D \sqrt{x^2 + y^2 + z^2}dxdydz$.

10. Let D denote the solid enclosed by the spheres $x^2 + y^2 + (z - 1)^2 = 1$ and $x^2 + y^2 + z^2 = 3$. Using spherical coordinates, set up iterated integrals that gives the volume of D.

Practice Problems 34: Hints/Solutions

1. The projection of the solid D on the xy-plane is given by $R = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, x^2 \le y \le x\}$. The solid D lies above the surface $z = f_1(x,y) = 0$ and below $z = f_2(x,y) = x$. Therefore $\iiint\limits_D y dx dy dz = \iint\limits_R \{\int_0^x y dz\} dx dy = \int_0^1 \int_{x^2}^x \int_0^x y dz dy dx$.

2. See Figure 1. Solving $4 - y^2 = 2 - y$ implies $y = -1, 2$. The projection of the solid D on the xy-plane is given by $R = [0,2] \times [-1,2]$. The solid lies above $z = f_1(x,y) = 2 - y$ and below $z = f_2(x,y) = 4 - y^2$. Therefore $\iiint\limits_D x dx dy dz = \iint\limits_R \{\int_{2-y}^{4-y^2} x dz\} dx dy = \int_0^2 \int_{-1}^2 \int_{2-y}^{4-y^2} x dz dy dx$.

3. (a) See Figure 2.

   (b) See Figure 3, Figure 4 and Figure 5.

   (c) $\int_0^4 \int_{\sqrt{x}}^2 \int_0^{2-y} dz dy dx = \int_0^2 \int_0^{y^2} \int_0^{2-y} dz dx dy = \int_0^2 \int_0^{2-z} \int_0^{y^2} dx dy dz = \int_0^2 \int_0^{2-y} \int_0^{y^2} dx dz dy$
   $= \int_0^4 \int_0^{2-\sqrt{x}} \int_{\sqrt{x}}^{2-z} dy dz dx = \int_0^2 \int_0^{(2-z)^2} \int_{\sqrt{x}}^{2-z} dy dx dz$.

4. Consider the change of variables $x = 2u, y = 4v$ and $z = 3w$. Note that the transformation $T(u,v,w) = (2u, 4v, 3w) = (x,y,z)$ maps E onto D and the Jacobian $J(u,v,w) = 24$.

5. (a) Solving $x^2 + y^2 = 36 - 3(x^2 + y^2)$ implies that $x^2 + y^2 = 9$. The projection of the solid D on the xy-plane is the circular disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 9\}$. The solid is bounded by $z = r^2$ and $z = 36 - 3r^2$. Therefore $D = \{(r,\theta,z) : 0 \le \theta \le 2\pi, 0 \le r \le 3, r^2 \le z \le 36 - 3r^2\}$.

   (b) The projection of D on the xy-plane is given by $\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \le 1\}$ which is described in cylindrical coordinates as $\{(r,\theta) : 0 \le \theta \le \pi, 0 \le r \le 2\sin\theta\}$. Therefore $D = \{(r,\theta,z) : 0 \le \theta \le \pi, 0 \le r \le 2\sin\theta, , 0 \le z \le r^2\}$.

   (c) The projection of the solid D on the xy-plane is the region between the circles $r = 1$ and $r = 3$. Allow θ to run from 0 to 2π and consider the cross section of the solid, perpendicular to the xy-plane, corresponding to a fixed θ. The cross section is a circle which is shown in Figure 6. The equation of the circle can be considered as $(r-2)^2 + z^2 = 1$ for $1 \le r \le 3$. Therefore $D = \{(r,\theta,z) : 0 \le \theta \le 2\pi, 1 \le r \le 3, 0 \le z \le \sqrt{1-(r-2)^2}\}$.

6. The projection of the solid D on the xy-plane is the circular disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. We will use the cylindrical coordinates. The solid D is bounded by $z = 0$ and $z = 2r$. Therefore $\iiint\limits_D x^2 dx dy dz = \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2\theta r dz dr d\theta$.

7. Note that $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 x dz dy dx = \iiint\limits_D x dx dy dz$ where D is the solid bounded below by $z = x^2 + y^2$ and above by $z = 4$. The projection of the solid on the xy-plane is given by $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$. By the cylindrical coordinates $\iiint\limits_D x dx dy dz = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\cos\theta r dz dr d\theta$.

8. (a) See Figure 7. The sphere $x^2+y^2+z^2=1$ is expressed as $\rho=1$ where as $x^2+y^2+(z-2)^2=4$ is expressed as $\rho=4\cos\phi$. The two spheres intersect at $\cos\phi=\frac{1}{4}$. For a fixed $\phi\in[0,\cos^{-1}\frac{1}{4}]$, ρ varies from 1 to $4\cos\phi$ in the given region. Therefore the region is given by $\{(\rho,\theta,\phi): 0\le\theta\le 2\pi,\ 0\le\phi\le\cos^{-1}\frac{1}{4},\ 1\le\rho\le 4\cos\phi\}$.

   (b) See Figure 8. The sphere is expressed as $\rho=\cos\phi$. The cone is expressed as $\rho\cos\phi=\rho\sin\phi$ that is $\phi=\frac{\pi}{4}$. For a fixed $\phi\in[0,\frac{\pi}{4}]$, ρ varies from 0 to $\cos\phi$ in the given region. Therefore the region is given by $\{(\rho,\theta,\phi): 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{4},\ 0\le\rho\le\cos\phi\}$.

   (c) See Figure 9. The cone is written as $\rho\cos\phi=\sqrt{3}\rho\sin\phi$; that is $\phi=\frac{\pi}{6}$. For a fixed $\phi\in[0,\frac{\pi}{6}]$, ρ varies from $\sec\phi$ to $2\sec\phi$ in the given region. Therefore the region is given by $\{(\rho,\theta,\phi): 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{6},\ \sec\phi\le\rho\le 2\sec\phi\}$.

9. See Figure 10. Let us use the spherical coordinates. The equation $z=\sqrt{x^2+y^2}$ is written as $\rho\cos\phi=\rho\sin\phi$. This implies that $\tan\phi=1$, i.e., $\phi=\frac{\pi}{4}$. The equation $z=4$ is written as $4=\rho\cos\phi$ that is $\rho=\frac{4}{\cos\phi}$. Therefore $\iiint\limits_D\sqrt{x^2+y^2+z^2}dxdydz=$
$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4\sec\phi}\rho\rho^2\sin\phi d\rho d\phi d\theta=2\pi 4^3\int_0^{\frac{\pi}{4}}\frac{\sin\phi}{\cos^4\phi}d\phi.$

10. See Figure 11. Solving $x^2+y^2+(z-1)^2=1$ and $x^2+y^2+z^2=3$ implies that $z=\frac{3}{2}$, i.e., $\rho\cos\phi=\frac{3}{2}$. The equation $x^2+y^2+(z-1)^2=1$ becomes $\rho=2\cos\phi$ in the spherical coordinates. The required volumes is the sum of the volume of the portion of the region $x^2+y^2+z^2\le 3$ that lies inside the cone $\phi=\frac{\pi}{6}$ and the volume of the portion of the region $x^2+y^2+(z-1)^2\le 1$ that lies inside the sphere $x^2+y^2+z^2=3$. Therefore the required volume is given by $\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{\sqrt{3}}\rho^2\sin\phi d\rho d\phi d\theta+\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{2\cos\phi}\rho^2\sin\phi d\rho d\phi d\theta.$