PP 35 : Parametric surfaces, surface area and surface integrals

1. Consider the surface (paraboloid) $z = x^2 + y^2 + 1$.

   (a) Parametrize the surface by considering it as a graph.

   (b) Show that $r(r, \theta) = (r\cos\theta, r\sin\theta, r^2 + 1), r \geq 0, 0 \leq \theta \leq 2\pi$ is a parametrization of the surface.

   (c) Parametrize the surface in the variables z and θ using the cylindrical coordinates.

2. For each of the following surfaces, describe the intersection of the surface and the plane $z = k$ for some $k > 0$; and the intersection of the surface and the plane $y = 0$. Further write the surfaces in parametrized form $r(z, \theta)$ using the cylindrical co-ordinates.

   (a) $4z = x^2 + 2y^2$ (paraboloid)
   (b) $z = \sqrt{x^2 + y^2}$ (cone)
   (c) $x^2 + y^2 + z^2 = 9, z \geq 0$ (Upper hemi-sphere)
   (d) $-\frac{x^2}{9} - \frac{y^2}{16} + z^2 = 1, z \geq 0$.

3. Let S denote the surface obtained by revolving the curve $z = 3 + \cos y, 0 \leq y \leq 2\pi$ about the y-axis. Find a parametrization of S.

4. Parametrize the part of the sphere $x^2 + y^2 + z^2 = 16, -2 \leq z \leq 2$ using the spherical co-ordinates.

5. Consider the circle $(y - 5)^2 + z^2 = 9, x = 0$. Let S be the surface (torus) obtained by revolving this circle about the z-axis. Find a parametric representation of S with the parameters θ and ϕ where θ and ϕ are described as follows. If (x, y, z) is any point on the surface then θ is the angle between the x-axis and the line joining $(0, 0, 0)$ and $(x, y, 0)$ and ϕ is the angle between the line joining (x, y, z) and the center of the moving circle (which contains (x, y, z)) with the xy-plane.

6. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$. Parametrize S by considering it as a graph and again by using the spherical coordinates.

7. Let S denote the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$. Find the area of S.

   (a) By considering S as a part of the graph $z = f(x, y)$ where $f(x, y) = 10 - 2x - 5y$.

   (b) By considering S as a parametric surface $r(u, v) = (u\cos v, u\sin v, 10 - u(2\cos v + 5\sin v))$, $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$.

8. Find the area of the surface $x = uv$, $y = u + v, z = u - v$ where $(u, v) \in D = \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\}$.

9. Find the area of the part of the surface $z = x^2 + y^2$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.

10. Let S be the hemisphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4, z \geq 0\}$.

    (a) Evaluate $\iint_S z^2 d\sigma$ by considering S as a graph: $z = f(x, y)$.

    (b) Evaluate $\iint_S z d\sigma$ by considering S as a parametric surface (but not as a graph).

11. Let S be the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant. Evaluate $\iint_S (z + 2xy) d\sigma$.

Practice Problems 35: Hints/Solutions

1. (a) $r(x, y) = (x, y, 1 + x^2 + y^2)$, $x, y \in \mathbb{R}$
   (b) Easy to verify.
   (c) $r(z, \theta) = (\sqrt{z-1}\cos\theta, \sqrt{z-1}\sin\theta, z)$, $z \geq 1, 0 \leq \theta \leq 2\pi$.

2. (a) For $z > 0$, $\frac{x^2}{4z} + \frac{y^2}{2z} = 1$. Hence $r(z, \theta) = (2\sqrt{z}\cos\theta, \sqrt{2z}\sin\theta, z), z \geq 0$ and $0 \leq \theta \leq 2\pi$.
   (b) $r(z, \theta) = (z\cos\theta, z\sin\theta, z), z \geq 0$ and $0 \leq \theta \leq 2\pi$.
   (c) $r(z, \theta) = (\sqrt{9 - z^2}\cos\theta, \sqrt{9 - z^2}\sin\theta, z), 0 \leq z \leq 3$ and $0 \leq \theta \leq 2\pi$.
   (d) $r(z, \theta) = (3\sqrt{z^2 - 1}\cos\theta, 4\sqrt{z^2 - 1}\sin\theta, z), z \geq 1$ and $0 \leq \theta \leq 2\pi$.

3. The intersection of the surface and the plane $y = t$ is a circle of radius $3 + \cos t$. The projection of this circle on the xz-plane is parametrized as $((3 + \cos t)\cos\theta, (3 + \cos t)\sin\theta), 0 \leq \theta \leq 2\pi$. Since t is varying from 0 to 2π, S is given by $r(t, \theta) = ((3 + \cos t)\cos\theta, t, (3 + \cos t)\sin\theta), 0 \leq t \leq 2\pi, 0 \leq \theta \leq 2\pi$.

4. The entire sphere is represented by $r(\theta, \phi) = (4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi)$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. To represent the given part, we apply $-2 \leq z \leq 2$. This implies $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$. Therefore the required parametrization is $r(\theta, \phi) = (4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi)$, $0 \leq \theta \leq 2\pi$ and $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$.

5. If $(x, y, z) \in S$ then $z = 3\sin\phi$ and $(x, y, 0) = (r\cos\theta, r\sin\theta, 0)$ where $r = 5 + 3\cos\phi$. Therefore a parametric representation is $r(\theta, \phi) = ((5+3\cos\phi)\cos\theta, (5+3\cos\phi)\sin\theta, 3\sin\phi)$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$.

6. The cone and the sphere intersect at the circle $x^2 + y^2 = 2, z = \sqrt{2}$. The surface S is given by $z = \sqrt{4 - x^2 - y^2}, x^2 + y^2 \leq 2$ and in spherical coordinates $x = 2\sin\phi\cos\theta, y = 2\sin\phi\sin\theta, z = 2\cos\phi$, $0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \theta \leq 2\pi$.

7. (a) The projection D of the surface on the xy-plane is $\{(x, y) : x^2+y^2 \leq 9\}$. The required area is $\iint_D \sqrt{1 + f_x^2 + f_y^2}dxdy = \iint_D \sqrt{1 + 4 + 25}dxdy = 9\sqrt{30}\pi$.
   (b) The area is $\int_0^3 \int_0^{2\pi} \sqrt{EG - F^2}dvdu$ where $\sqrt{EG - F^2} = |r_u \times r_v| = u\sqrt{30}$.

8. The surface is $r(u, v) = (uv, u + v, u - v)$ and hence $\sqrt{EG - F^2} = \sqrt{4 + 2(u^2 + v^2)}$. Therefore the required area is $\iint_D \sqrt{4 + 2(u^2 + v^2)}dudv = \int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2}rdrd\theta$.

9. The entire surface $z = x^2 + y^2$ is parametrized as $r(r, \theta) = (r\cos\theta, r\sin\theta, r^2), r \geq 0$ and $0 \leq \theta \leq 2\pi$. Now $\sqrt{EG - F^2} = |r_\theta \times r_r| = r\sqrt{4r^2 + 1}$. Since the projection of the part of the surface on the xy-plane is the region between $x^2 + y^2 = 4$ and $x^2 + y^2 = 4$, $2 \leq r \leq 4$. Therefore the required area is $\int_0^{2\pi} \int_2^4 r\sqrt{4r^2 + 1}drd\theta$.

10. (a) Since $2x + 2zz_x = 0$, $z_x = -\frac{x}{z}$. Similarly $z_y = -\frac{y}{z}$. Hence $\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} = \frac{2}{z}$. The projection of the S on the xy-plane is $D = \{(x, y) : x^2+y^2 \leq 4\}$. Therefore $\iint_S z^2 d\sigma = \iint_D z^2 \frac{2}{z}dxdy = 2\iint_D \sqrt{4 - x^2 - y^2}dxdy = 2\int_0^{2\pi} \int_0^2 \sqrt{4 - r^2}rdrd\theta$.
    (b) The surface is given by $x = 2\cos\theta\sin\phi, y = 2\sin\theta\sin\phi, z = 2\cos\phi$ where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$. Sine $\sqrt{EG - F^2} = 4\sin\phi$, $\iint_S d\sigma = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2\cos\phi 4\sin\phi d\phi d\theta$.

11. The surface is $r(x, \theta) = (x, \cos\theta, \sin\theta), 0 \leq x \leq 3$ and $0 \leq \theta \leq \frac{\pi}{2}$. Therefore $\sqrt{EG - F^2} = |r_x \times r_\theta| = 1$. Hence $\iint_S (z+2xy) = \int_0^{\frac{\pi}{2}} \int_0^3 (\sin\theta + 2x\cos\theta)(1)dxd\theta = \int_0^{\frac{\pi}{2}} (3\sin\theta + 9\cos\theta d\theta$.