Practice Problems 7 : Mean Value Theorem, Cauchy Mean Value Theorem, L'Hospital Rule

1. Use the mean value theorem (MVT) to establish the following inequalities.
   (a) $e^x \geq 1 + x$ for $x \in \mathbb{R}$.
   (b) $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$.
   (c) $\frac{x-1}{x} < lnx < x - 1$ for $x > 1$.

2. Does there exist a differentiable function $f : [0, 2] \to \mathbb{R}$ satisfying $f(0) = -1, f(2) = 4$ and $f'(x) \leq 2$ for all $x \in [0, 2]$?

3. Let $f : [0, 1] \to \mathbb{R}$ be differentiable such that $|f'(x)| < 1$ for all $x \in [0, 1]$. Show that there exists at most one $c \in [0, 1]$ such that $f(c) = c$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that, for some $\alpha \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1$ for all $x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$. Show that the sequence (a_n) converges.

5. Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0, f(0))$ and $(1, f(1))$ intersect the graph of f at a point $(a, f(a))$ where $0 < a < 1$. Show that there exists $x_0 \in [0, 1]$ such that $f''(x_0) = 0$.

6. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose that f is differentiable on $(0, 1)$ and $\lim_{x\to 0} f'(x) = \alpha$ for some $\alpha \in \mathbb{R}$. Show that $f'(0)$ exists and $f'(0) = \alpha$.

7. Let $f : [0, 1] \to \mathbb{R}$ be differentiable and $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)|$ for all $x \in [0, 1]$. Show that $f(x) = 0$ for all $x \in [0, 1]$.

8. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and $f(0) = 0$. Suppose that $f'(x)$ exists for all $x \in (0, \infty)$ and f' is increasing on $(0, \infty)$. Show that the function $g(x) = \frac{f(x)}{x}$ is increasing on $(0, \infty)$.

9. Establish the following inequalities.
   (a) For $\alpha > 1$, $(1 + x)^\alpha \geq 1 + \alpha x$ for all $x > -1$.
   (b) For $x > 0$, $e \ln x \leq x$.

10. Let $f : [a, b] \to \mathbb{R}$ be differentiable and $a \geq 0$. Using Cauchy mean value theorem, show that there exist $c_1, c_2 \in (a, b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists at some $c \in \mathbb{R}$. Using L'Hospital rule, show that

$$\lim_{h\to 0} \frac{f(c + h) - 2f(c) + f(c - h)}{h^2} = f''(c).$$

    Show with an example that if the above limit exists then $f''(c)$ may not exist.

12. **(*)** Let $f : [a, b] \to \mathbb{R}$ be differentiable. If $f'(x) \neq 0$ for all $x \in [a, b]$, then show that either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$.

13. **(*)** Let $f : [a, b] \to \mathbb{R}$ be differentiable and $\alpha \in \mathbb{R}$ be such that $f'(a) < \alpha < f'(b)$. Define $g(x) = f(x) - \alpha x$ for all $x \in [a, b]$.
    (a) Using the fact that $g'(a) < 0$ and $g'(b) > 0$, show that the condition $g'(x) \neq 0$ for all $x \in [a, b]$ leads to a contradiction.
    (b) Show that there exists $c \in [a, b]$ such that $f'(c) = \alpha$.
    (c) From (b), conclude that if a function f is differentiable at every point of an interval $[a, b]$, then its derivative f' has the IVP on $[a, b]$.

Practice Problems 7: Hints/Solutions

1. (a) Let $x > 0$. By the MVT there exists $c \in (0, x)$ such that $e^x - e^0 = e^c(x - 0)$. This implies that $e^x \geq 1 + x$. The proof is similar for the case $x < 0$.

   (b) By the MVT, for $f(x) = \sqrt{x}$, there exists $c \in (n, n+1)$ such that $\sqrt{n+1} - \sqrt{n} = \frac{1}{2\sqrt{c}}$.

   (c) By the MVT, there exists $c \in (1, x)$ such that $\ln x - \ln 1 = \frac{1}{c}(x - 1)$.

2. If so, then by the MVT there exits $c \in (0, 2)$ such that $5 = f(2) - f(0) = 2f'(c)$.

3. Suppose $f(c_1) = c_1$ and $f(c_2) = c_2$ for some $c_1, c_2 \in [0, 1]$ and $c_1 \neq c_2$. Then by the MVT, there exists $c_0 \in (0, 1)$ such that $c_2 - c_1 = f(c_2) - f(c_1) = f'(c_0)(c_2 - c_1)$; i.e., $f'(c_0) = 1$.

4. Note that, for some c, $|a_{n+2} - a_{n+1}| = |f(a_{n+1}) - f(a_n)| = |f'(c)||a_{n+1} - a_n| < \alpha|a_{n+1} - a_n|$. The sequence satisfies the Cauchy criterion and hence it converges.

5. Using the MVT on $[0, a]$ and $[a, 1]$, obtain $b \in (0, a)$ and $c \in (a, 1)$ such that $\frac{f(a)-f(0)}{a-0} = f'(b)$ and $\frac{f(1)-f(a)}{1-a} = f'(c)$. Note that $f'(b) = f'(c)$ because they are slopes of the same chord. By Rolle's theorem there exists $x_0 \in (b, c)$ such that $f''(x_0) = 0$.

6. For every $x > 0$, by the MVT, there exists $c_x \in (0, x)$ such that $\frac{f(x)-f(0)}{x} = f'(c_x)$. Now $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} f'(c_x) = \lim_{c_x \to 0} f'(c_x) = \alpha$.

7. For $x \in (0, 1)$, by the MVT, there exists x_1 such that $0 < x_1 < x$ and $f(x) = f'(x_1)x$. This implies that $|f(x)| \leq x|f(x_1)|$. Similarly there exists x_2 such that $0 < x_2 < x_1$ and $|f(x_1)| \leq x_1|f(x_2)|$. Therefore $|f(x)| \leq x^2|f(x_2)|$. Find a sequence (x_n) in $(0, 1)$ such that $|f(x)| \leq x^n|f(x_n)|$. Since f is bounded on $[0, 1]$, $x^n|f(x_n)| \to 0$. Hence $f(x) = 0$.

8. Note that $g'(x) = \frac{xf'(x)-f(x)}{x^2} = \frac{f'(x)-\frac{f(x)}{x}}{x}$. Observe that, by the MVT, $\frac{f(x)}{x} = f'(c_x)$ for some $c_x \in (0, x)$. Since f' is increasing, $g'(x) \geq 0$. Hence g is increasing.

9. (a) Let $\alpha > 1$ and $f(x) = (1 + x)^\alpha - (1 + \alpha x)$ on $(-1, \infty)$. Therefore $f'(x) \leq 0$ on $(-1, 0]$ and $f'(x) \geq 0$ on $[0, \infty)$. Hence $f(x) \geq f(0) = 0$ on $(-1, 0]$ and $f(x) \geq f(0) = 0$ on $[0, \infty)$. Therefore $f(x) \geq 0$ on $(-1, \infty)$.

   (b) Define $f(x) = x - e \ln x$ on $(0, \infty)$. Then $f'(x) = \frac{x-e}{x}$. Therefore $f'(x) > 0$ on (e, ∞) and $f'(x) < 0$ on $(0, e)$. Hence $f(x) > f(e)$ for all $x \in (0, \infty)$ and $x \neq e$.

10. Apply Cauchy MVT to $f(x)$ and $g_1(x) = x$. Again apply to $f(x)$ and $g_2(x) = x^2$.

11. Since $f''(c)$ exists there exists a $\delta > 0$ such that $f'(x)$ exists on $(c - \delta, c + \delta)$. Therefore by L'Hospital rule, the given limit is equal to $\lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{h^2}$ if it exists. But $\lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{2h} = \frac{1}{2}\left[\lim_{h\to 0} \frac{f'(c+h)-f'(c)}{h} + \lim_{h\to 0} \frac{f'(c-h)-f'(c)}{-h}\right] = \frac{1}{2}[f''(c) + f''(c)]$.

    Let $f(x) = 1$ on $(0, \infty)$, $f(0) = 0$ and $f(x) = -1$ on $(-\infty, 0)$. Then f is not continuous at 0 hence $f''(0)$ does not exist. It can be easily verified that the limit given in the question exists.

12. Since f is one-one, it either strictly increasing or strictly decreasing (see Problem 15 of Practice Problems 5). Apply the definition of f' to show that either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$.

13. (a) Follows from Problem 12.

    (b) Trivial.

    (c) Trivial.