

Practice Problems 8 : Fixed point iteration method and Newton's method

1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable and $\alpha \in \mathbb{R}$ be such that $|g'(x)| \leq \alpha < 1$ for all $x \in \mathbb{R}$.
 - (a) Show that the sequence generated by the fixed point iteration method for g converges to a fixed point of g for any starting value $x_0 \in \mathbb{R}$.
 - (b) Show that g has a unique fixed point.
2. Let $x_0 \in \mathbb{R}$. Using the fixed point iteration method generate a sequence of approximate solutions of the equation $x - \frac{1}{2} \sin x = 1$ for the starting value x_0 .
3. Let $g : [0, 1] \rightarrow [0, 1]$ be defined by $g(x) = \frac{1}{1+x^2}$. Let (x_n) be the sequence generated by the fixed point iteration method for g with the starting value $x_0 = 1$. Show that (x_n) converges.
4. Let f be a differentiable function on \mathbb{R} (or on $[a, b]$). Suppose that $F(x) = x - \frac{f(x)}{f'(x)}$ is defined for all $x \in \mathbb{R}$ (or for all $x \in [a, b]$).
 - (a) Show that a point $\bar{x} \in \mathbb{R}$ (or $\bar{x} \in [a, b]$) is a solution of the equation $f(x) = 0$ if and only if \bar{x} is a fixed point of F .
 - (b) Let $x_0 \in \mathbb{R}$ (or $x_0 \in [a, b]$) and $x_{n+1} = F(x_n), n = 0, 1, 2, \dots$. Show that if $(f'(x_n))$ is bounded and (x_n) converges to some \bar{x} then $f(\bar{x}) = 0$.
5. Let $f(x) = e^{-\frac{1}{x^2}}$ if $x \neq 0$ and $f(0) = 0$. Suppose that $0 < x_0 < 1$ and (x_n) be the sequence generated by Newton's method with the starting value x_0 . Show that (x_n) converges.
6. Let $f(x) = 3x^{\frac{1}{3}}$. Let $x_0 > 0$ and (x_n) be the sequence generated by Newton's method. Show that (x_n) oscillates and is unbounded.
7. Let $f : [-10, 10] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x-1} & \text{if } x \geq 1 \\ -\sqrt{1-x} & \text{if } x < 1. \end{cases}$$

Let $x_0 \neq 1$ and (x_n) be the sequence generated by Newton's method with the starting value x_0 . Show that $x_n = x_0$ if n is even and $x_n = 2 - x_0$ if n is odd.

8. Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable and $f'(x) \neq 0$ for all $x \in [a, b]$. Define F by $F(x) = x - \frac{f(x)}{f'(x)}$ for all $x \in [a, b]$. Suppose that $F(x) \in [a, b]$ for all $x \in [a, b]$.
 - (a) Suppose that f'' exists and for all $x \in [a, b]$ and

$$\left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| \leq \alpha < 1$$

for some $\alpha \in \mathbb{R}$. Show that the sequence (x_n) , generated by Newton's Algorithm i.e., defined by $x_{n+1} = F(x_n)$, converges for any starting value $x_0 \in [a, b]$.

- (b) Let $f(x) = (x-1)^2$ and $x_0 \in [0, 2]$. Show that the sequence generated by Newton's method converges to 1 for the starting value x_0 .
- (c) Let $f(x) = x^2 - 7$ and $x_0 \in [2, 7]$. Show that the sequence generated by Newton's method converges to $\sqrt{7}$ for the starting value x_0 .

9. (*) Let $f : [a, b] \rightarrow [a, b]$ be continuous and ℓ be a fixed point of f . Suppose that f is differentiable on (a, b) and $|f'(x)| < 1$ for all $x \in (a, b)$. Let $x_0 \in [a, b]$ and $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \dots$

(a) Show that f has a unique fixed point.

(b) Show that $|x_{n+1} - \ell| \leq |x_n - \ell|$ for all $n \in \mathbb{N}$.

(c) If (x_{n_k}) is a subsequence of (x_n) , show that $|x_{n_{k+1}} - \ell| \leq |x_{n_k+1} - \ell| \leq |x_{n_k} - \ell|$ for all $k \in \mathbb{N}$.

(d) If a subsequence (x_{n_k}) of (x_n) converges to some x_0 , show that $x_0 = \ell$.

(e) Show that $x_n \rightarrow \ell$.

(f) Show that for $f(x) = \frac{x^2}{2}$, $a = 0$ and $b = 1$ the sequence (x_n) converges.

Practice Problems 8: Hints/Solutions

1. (a) Suppose $x_{n+1} = g(x_n)$. Then, by Problem 4 of Practice Problem 7, (x_n) converges. If $x_n \rightarrow \bar{x}$, then by continuity of g , $\bar{x} = g(\bar{x})$.
 (b) There is a fixed point of g by (a). For uniqueness see the solution of Problem 3 of Practice Problems 7.
2. Write $x = g(x)$ where $g(x) = 1 + \frac{1}{2} \sin x$ and note that $|g'(x)| \leq \frac{1}{2} < 1$ for all $x \in \mathbb{R}$. By Problem 1, the sequence (x_n) defined by $x_{n+1} = g(x_n)$ converges to a fixed point of g . Since a fixed point of g is a solution of $x - \frac{1}{2} \sin x = 1$, the elements x_n 's are approximate solutions.
3. Observe that $g : [0, 1] \rightarrow [0, 1]$ and $|g'(x)| = \frac{2x}{(1+x^2)^2}$ achieves its maximum at $x = \frac{1}{\sqrt{3}}$ on $[0, 1]$. Therefore $|g'(x)| \leq \frac{9}{8\sqrt{3}} < 1$ for all $x \in [0, 1]$. Hence by Theorem 8.1, the sequence (x_n) converges.
4. (a) Trivial.
 (b) Note that $f'(x_n)(x_{n+1} - x_n) = f(x_n)$. If $x_n \rightarrow \bar{x}$ then $f(x_n) \rightarrow 0$. Hence $f(\bar{x}) = 0$.
5. For f , $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3}{2}$. Then (x_n) is decreasing and bounded below.
6. In this case, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n$. Therefore (x_n) is unbounded.
7. For given f , $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2 - x_n$. This implies the answer.
8. (a) Observe that $F'(x) = \left| \frac{f(x)f''(x)}{[f'(x)]^2} \right|$. Apply Theorem 8.1.
 (b) The function $F(x) = x - \frac{f(x)}{f'(x)} = \frac{1}{2}(x+1)$ and $F : [0, 2] \rightarrow [0, 2]$. Moreover $|F'(x)| = \left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| = \frac{1}{2} = \alpha < 1 \forall x \in [0, 2]$. So the problem follows from (a) and Problem 4.
 (c) The function $F(x) = x - \frac{f(x)}{f'(x)} = \frac{x}{2} + \frac{7}{2x}$. It is shown in Problem 9 of Practice Problems 2 that the sequence (x_n) defined by $x_{n+1} = \frac{1}{2} \left(x_n + \frac{7}{x_n} \right)$, converges.

The problem can also be solved using (a) as follows. By finding the maximum and minimum values of the function $F(x)$ on $[2, 7]$ or otherwise, verify that $F : [2, 7] \rightarrow [2, 7]$. Again by finding the maximum and minimum values of the function $F'(x)$ or otherwise, verify that $|F'(x)| = \left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| \leq \frac{3}{7}$ for all $x \in [2, 7]$. Therefore the problem follows from (a) and Problem 4.

9. (a) See the solution of Problem 3 of Practice Problems 7 for uniqueness.
 (b) By the mean value theorem $|x_{n+1} - \ell| = |f(x_n) - f(\ell)| < |x_n - \ell|$.
 (c) This follows from (b) and the definition of a subsequence.
 (d) Suppose $x_{n_k} \rightarrow x_0$ and $x_0 \neq \ell$. Then $|x_{n_{k+1}} - \ell| \rightarrow |x_0 - \ell|$ and $|x_{n_k} - \ell| \rightarrow |x_0 - \ell|$. It follows from (c) that $|x_{n_{k+1}} - \ell| \rightarrow |x_0 - \ell|$; i.e., $|f(x_{n_k}) - f(\ell)| \rightarrow |x_0 - \ell| = |f(x_0) - f(\ell)|$. But by the mean value theorem $|f(x_0) - f(\ell)| < |x_0 - \ell|$ which is a contradiction.
 (e) Follows from (d), Bolzano-Weierstrass Theorem and Problem 8 of Practice Problems 3.
 (f) It is easily seen that $f : [0, 1] \rightarrow [0, 1]$ and $|f'(x)| < 1$ for all $x \in (0, 1)$. So by (e), (x_n) converges.