Practice Problems 9 : Tests for maxima and minima, Curve sketching

1. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = f(x)g(x)$ where f and g are non-negative functions. Show that h has a local maximum at a if f and g have a local maximum at a.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (\sin x - \cos x)^2$. Find the maximum value of f on $\mathbb{R}$.

3. Let $f : [-2, 0] \to \mathbb{R}$ be defined by $f(x) = 2x^3 + 2x^2 - 2x - 1$. Find the maximum and minimum values of f on $[-2, 0]$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'(x) = 14(x-2)(x-3)^2(x-4)^3(x-5)^4,\ x \in \mathbb{R}$. Find all the points of local maxima and local minima.

5. Let $x_1, x_2, ..., x_n \in \mathbb{R}$ and $f(x) = \sqrt{(x-x_1)^2 + (x-x_2)^2 + ... + (x-x_n)^2},\ x \in \mathbb{R}$. Find the point of absolute minimum of the function f.

6. Find the points of local maxima and local minima of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^4 e^{-x^2}$.

7. (a) Let $\alpha \in \mathbb{R}$. Among all positive real numbers x and y satisfying $x + y = \alpha$, show that the product xy is largest when $x = y = \frac{\alpha}{2}$.
   (b) Among all rectangles of given perimeter, show that the square has the largest area.

8. (a) Find the point of absolute maximum of the function $f(x) = x^{\frac{1}{x}}$ for $x > 0$.
   (b) Show that $e^\pi > \pi^e$.

9. (a) Show that $\frac{\ln a}{a} > \frac{\ln b}{b}$ when $b > a > e$.
   (b) For $b > a > e$, show that $a^b > b^a$.

10. (a) For $x \geq 0$ and $0 \leq p \leq 1$, show that $(1+x)^p \leq 1 + x^p$.
    (b) Show that $(a+b)^p \leq a^p + b^p$ for all $0 \leq p \leq 1$ and $a, b > 0$.

11. An open-top box with square base is to be made. The volume of the box should be 13500 cm^2. Find the width and height of the box that minimize the amount of material to be used.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with the following properties:
    $f(-1) = 4,\quad f(0) = 2,\quad f(1) = 0,\quad f'(x) > 0$ for $|x| > 1,\quad f'(x) < 0$ for $|x| < 1$,
    $f'(1) = 0,\quad f'(-1) = 0,\quad f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.
    Sketch the graph of f.

13. Sketch the graphs of the following functions after finding the intervals of decrease/increase, intervals of concavity/convexity, points of local minima/local maxima, points of inflection and asymptotes.
    (a) $f(x) = \frac{x^2+x-5}{x-1}$ (b) $f(x) = \frac{2x^2-1}{x^2-1}$ (c) $f(x) = \frac{x^2}{x^2+1}$
    (d) $f(x) = \frac{2x^3}{x^2-4}$ (e) $f(x) = 3x^4 - 8x^3 + 12$.

14. (a) Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{x^2}{x^3+200}$. Find the point of maximum of f in $(0, \infty)$.
    (b) Let (a_n) be a sequence defined by $a_n = \frac{n^2}{n^3+200},\ n \in \mathbb{N}$. Show that the largest term of the sequence (a_n) is a_7.

15. **(*)** Let $f(x) = (x+1)\ln(x+1) - x\ln x - \ln(2x+1)$ for $x > 0$. Show that f is strictly increasing on $(0, \infty)$. Further, show that the sequence $\left(\frac{(n+1)^{n+1}}{n^n(2n+1)}\right)$ is strictly increasing.

Practice Problems 9: Hints/Solutions

1. Find $\delta_1 > 0$ such that $f(a) \geq f(x)\ \forall\ x \in (a-\delta_1, a+\delta_1)$ and $\delta_2 > 0$ such that $g(a) \geq g(x)\ \forall\ x \in (a-\delta_2, a+\delta_2)$. Then $h(a) \geq h(x)\ \forall\ x \in (a-\delta, a+\delta)$ for $\delta = \min\{\delta_1, \delta_2\}$.

2. Since $f(x+2\pi) = f(x)\ \forall\ x \in \mathbb{R}$, i.e., f is periodic with period 2π, $\sup\{f(x) : x \in \mathbb{R}\} = \sup\{f(x) : x \in [0, 2\pi]\}$. Note that, on $(0, 2\pi)$, $f'(x) = 0$ at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$. Since f achieves its supremum on $[0, 2\pi]$, the greatest value among the points mentioned above and the end points 0 and 2π is the maximum value of the function. Comparing the values of f at these points, we find that the maximum value of f is 2.

3. Note that, on $(-2, 0)$, $f'(x) = 0$ only at $x = -1$. Comparing the values of f at $x = -1$ and the end points -2 and 0, we find that the maximum value of f is 1 and the minimum value is -5.

4. Observe that f' changes sign from positive to negative at $x = 2$ and negative to positive at $x = 4$. The local maximum is $x = 2$ and local minimum is $x = 4$.

5. Let $g(x) = (x - x_1)^2 + (x - x_2)^2 + ... + (x - x_n)^2$. Note that the point of minimum of f and g are same. At $x = \frac{x_1+x_2+...+x_n}{n}$, $g'(x) = 0$ and $g''(x) = 2n > 0$. Therefore the point of minimum of f is $\frac{x_1+x_2+...+x_n}{n}$.

6. Since $f'(x) = 2x^3e^{-x^2}(4-2x^2)$, f' vanishes at $x = -\sqrt{2}, \sqrt{2}, 0$. The sign of f' change from $+$ to $-$ at $x = -\sqrt{2}$, from $+$ to $-$ at $x = \sqrt{2}$ and from $-$ to $+$ at $x = 0$. Therefore $\sqrt{2}$ and $-\sqrt{2}$ are points of local maxima and 0 is a point of local minimum.

7. (a) If $x + y = \alpha$ then $xy = x(\alpha - x)$. So let $f(x) = \alpha x - x^2$. Then $x = \frac{\alpha}{2}$ is the point of maximum of f.

   (b) Let α be the perimeter and x and y denote the lengths of the sides of the rectangle. Then $x + y = \frac{\alpha}{2}$. The area is xy which is maximum when $x = y$ by (a).

8. (a) The derivative $f'(x) = x^{\frac{1}{x}}\frac{1-\ln x}{x^2}$ vanishes only at $x = e$. Since the sign of f' changes from positive to negative at $x = e$, the point of maximum is $x = e$.

   (b) By (a), $f(e) = e^{\frac{1}{e}} > f(\pi) = \pi^{\frac{1}{\pi}}$. Therefore $\left(e^{\frac{1}{e}}\right)^{e\pi} > \left(\pi^{\frac{1}{\pi}}\right)^{e\pi}$.

9. (a) Let $f(x) = \frac{\ln x}{x}$ for $x > 0$. Because $f'(x) = \frac{1-\ln x}{x^2} < 0$ for $x > e$, f is decreasing on (e, ∞). Therefore $\frac{\ln a}{a} > \frac{\ln b}{b}$ when $b > a > e$.

   (b) For $b > a > e$, by (a), $b \ln a > a \ln b$. This implies that $e^{b\ln a} > e^{a\ln b}$; i.e, $e^{\ln a^b} > e^{\ln b^a}$.

10. (a) Let $f(x) = 1 + x^p - (1+x)^p$ for $x \geq 0$. Then $f'(x) = p\left[\frac{1}{x^{1-p}} - \frac{1}{(1+x)^{1-p}}\right] > 0$ for all $x > 0$. This implies that $f(x) > f(0) = 0$ for $x > 0$.

    (b) It is sufficient to show that $(\frac{a}{b} + 1)^p \leq (\frac{a}{b})^p + 1$ which follows from (a).

11. Let x be the width of the square base. Then the height of the box is $\frac{13500}{x^2}$. Therefore the surface area is $S(x) = x^2 + 4\frac{13500}{x}$. Hence $x = 30$ is the point of minimum of S.

12. See Figure 1 for the graph of f.

13. (a) Note that $f(x) = x+2-\frac{3}{x-1}$, $f'(x) = 1+\frac{3}{(x-1)^2}$ and $f''(x) = \frac{-6}{(x-1)^3}$. The asymptotes are $x = 1$ and $y = x + 2$. The function is increasing on $(-\infty, 1)$ and $(1, \infty)$. The function is convex for $x < 1$ and concave for $x > 1$. The function has no point of inflection (note that f is not defined at $x = 1$). There is no point of local maximum and local minimum. The graph of f is given in Figure 2.

(b) Observe that $f(x) = 2+\frac{1}{x^2-1}$, $f'(x) = \frac{-2x}{(x^2-1)^2}$ and $f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$. The asymptotes are $x = 1$, $x = -1$ and $y = 2$. The function is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$. The point of local maximum is 0. The function is convex on $(-\infty, 1)$ and $(1, \infty)$ and concave on $(-1, 1)$. There is no point of inflection. See Figure 3 for the graph.

(c) We have $f(x) = 1 - \frac{1}{x^2+1}$, $f'(x) = \frac{2x}{(x^2+1)^2}$ and $f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$. The asymptote is $y = 1$. The function is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. The function is concave on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$; and convex on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The points of inflection are $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$. The function has local minimum at $x = 0$. For the graph see Figure 4.

(d) Note that $f(x) = 2x+\frac{8x}{x^2-4}$, $f'(x) = \frac{2x^2(x^2-12)}{(x^2-4)^2}$ and $f''(x) = \frac{16x(x^2+12)}{(x^2-4)^3}$. Then $x = 2$, $x = -2$ and $y = 2x$ are the asymptotes. The function is increasing on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$. The function is decreasing on $(-2\sqrt{3}, -2)$, $(-2, 2)$ and $(2, 2\sqrt{3})$. The function is convex on $(-2, 0)$ and $(2, \infty)$ and concave on $(-\infty, -2)$ and $(0, 2)$. The point of inflection is 0. See Figure 5 for the graph.

(e) Here $f'(x) = 12x^2(x - 2)$ and $f''(x) = 12x(3x - 4)$. Therefore f is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$. There is no asymptote. The point of local minimum is 2. The function is convex on $(-\infty, 0)$ and $(\frac{4}{3}, \infty)$ and concave on $(0, \frac{4}{3})$. The points of inflections are 0 and $\frac{4}{3}$. See the graph in Figure 6.

14. (a) Since $f'(x) = \frac{x(400-x^3)}{(x^3+200)^2}$, f is increasing on $(0, 400^{\frac{1}{3}})$ and decreasing on $(400^{\frac{1}{3}}, \infty)$. Therefore, the point of maximum is $400^{\frac{1}{3}}$.

(b) We will use (a). Note that $7 < 400^{\frac{1}{3}} < 8$. Thus the largest term of the sequence can be either a_7 or a_8. But $a_7 = \frac{49}{543} > a_8 = \frac{8}{89}$. Therefore a_7 is the largest term.

15. Note that $f'(x) = \ln(x + 1) - \ln x - \frac{2}{2x+1}$ and $f''(x) = \frac{1}{x+1} - \frac{1}{x} + \frac{4}{(2x+1)^2}$. Since $f''(x) < 0$ on $(0, \infty)$, f' is decreasing. Write $f'(x) = \ln(1 + \frac{1}{x}) - \frac{2}{2x+1}$ and observe that $f'(x) \to 0$ as $x \to \infty$. Therefore $f'(x) > 0$ for all $x > 0$. It is easy to see that $\ln a_n = f(n)$. Since $f(n + 1) > f(n)$, $\ln(a_{n+1}) > \ln(a_n)$. Therefore $e^{\ln(a_{n+1})} > e^{\ln(a_n)}$ and hence $a_{n+1} > a_n$.