MTH102N
ASSIGNMENT–LA 2

(1) Find all elements of S_3 (the set of all permutations of the set $\{1,2,3\}$) and determine which permutations are odd.

(2) Let $\sigma \in S_4$ be given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

(a) Find sign of σ and sign of σ^{-1}.
(b) What does $\sigma^2 := \sigma \circ \sigma$ do to $(1,2,3,4,5)$?

(3) Find two 2×2 invertible matrices A and B such that $A \neq cB$, for any scalar c and $A+B$ is not invertible.

(4) Show that an $n \times n$ matrix A is invertible iff the system $AX = Y$ has a solution for every $Y = (y_1, \ldots, y_n)^t$.

(5) Let $A = [a_{ij}]$ be an invertible matrix and let $B = [p^{i-j}a_{ij}]$. Find the inverse of B also find $|B|$.

(6) Let A be an $n \times n$ matrix. Show that $|A| = 0$ iff there exist $x_1, \ldots, x_n$, not all zero, such that $A(x_1, \ldots, x_n)^t = 0$.

(7) Find the determinant of

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

(8) Find, by definition, the determinant of $A = [a_{ij}]$ in each of the following cases:
(a) A is a diagonal matrix
(b) A is a lower triangular matrix (i.e. $a_{ij} = 0$ for all $j > i$)
(c) A is an upper triangular matrix (i.e. $a_{ij} = 0$ for all $j < i$)

(9) For an $n \times n$ matrix $A = [a_{ij}]$ prove that $|A| = |A^t|$.

(10) The numbers $1375, 1287, 4191$ and 5731 are all divisible by 11. Prove that the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 5 & 7 & 1 & 1 \end{bmatrix}$$

is also divisible by 11.

(11) Find the determinant of

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 2 & 3 & 4 & \dots & n \\ 3 & 3 & 3 & 4 & \dots & n \\ .. & .. & .. & .. & \dots & n \\ n & n & n & n & \dots & n \end{bmatrix}.$$

(12) For a complex matrix $A = [a_{ij}]$, let $\bar{A} = [\ \overline{a_{ij}}\ ]$ and $A^* = \bar{A}^t$. Show that $|\bar{A}| = |A^*| = \overline{|A|}$. Therefore if A is Hermitian (that is $A^* = A$) then its determinant is real.

(13) A real matrix A is said to be orthogonal if $AA^t = I$. Show that if A is orthogonal then $|A| = \pm 1$.

(14) Let A be an invertible square matrix with integer entries. Show that A^{-1} has integer entries if and only if $|A| = \pm 1$.