MTH102N
ASSIGNMENT–LA 5

(1) Let C be an $m \times n$ matrix and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the linear transformation defined by C. Show that the matrix of T with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is C.

(2) Let the linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $T(x, y) = (ax + by, cx + dy)$. Find the matrix of T with respect to the standard basis of $\mathbb{R}^2$. Now do the same by considering the basis $\{(0, 1), (1, 0)\}$ on domain and range of T.

(3) Consider the linear map $T : \mathbb{C} \rightarrow \mathbb{C}$ given by $T(z) = iz$. By considering the basis $\{1, i\}$ of $\mathbb{C}$ (over $\mathbb{R}$) on domain and codomain of T find the matrix of T.

(4) Let $T : V \rightarrow V$ be a linear map such that $\text{Ker}(T) = \text{Range}(T)$. What can you say about T^2. On $\mathbb{R}^2$ can you give example of such a map?

(5) Does there exist a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ such that $\text{Range}(T) = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}$?

(6) Let V be a vector space of dimension n and let $A = \{v_1, \ldots, v_n\}$ be an ordered basis of V. Suppose $w_1, \ldots, w_n \in V$ and let $(a_{1j}, \ldots, a_{nj})^t$ denote the coordinates of w_j with respect to A. Put $C = [a_{ij}]$.

Then show that $w_1, \ldots, w_n$ is a basis of V if and only if C is invertible.

(7) Find the range and kernel of $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by $T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$.

(8) Let T be a linear transformation from an n dimensional vector space V to an m dimensional vector space W and let C be the matrix of T with respect to a basis A of V and B of W. Show that (a) $\rho(T) = \text{rank}(C)$; (b) T is one-one if and only if $\text{rank}(C) = n$; (c) T is onto if and only if $\text{rank}(C) = m$; (d) T is an isomorphism (that is, one-one and onto) if and only if $m = \text{rank}(C) = n$.

(9) Let $<, >$ be any inner product on $\mathbb{R}^n$. Show that $< x, y >= x^t Ay$ for all vectors $x, y \in \mathbb{R}^n$ where A is the symmetric $n \times n$ matrix whose (i, j)th entry is $< e_i, e_j >$, the vector e_i being the standard basis vectors of $\mathbb{R}^n$.

(10) Show that the norm of a vector in a vector space V has the following three properties

(a) $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda|\|v\|$ for all $\lambda \in \mathbb{R}$ and $v \in V$.

(c) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

(11) Use Gram-Schmidt process to transform each of the following into an orthonormal basis;

(a) $\{(1,1,1),(1,0,1),(0,1,2)\}$ for $\mathbb{R}^3$ with dot product.

(b) Same set as in (i) but using the inner product defined by $<(x,y,z),(x',y',z')>= xx' + 2yy' + 3zz'$.

(12) Let U be a proper subspace of the inner product space V. Let $U^{\perp} = \{v \in V : < v, u >= 0 \ \ \forall u \in U\}$. Show that $U^{\perp}$ is a subspace of V (it is called *orthogonal complement* of U). Let $U = \{\alpha(1,2,3) : \alpha \in \mathbb{R}\}$ be a subspace of $\mathbb{R}^3$ with scalar product. Find $U^{\perp}$. Also, show that $S^{\perp}$ is a subspace of V for any arbitary subset S of V.

(13) Let U_1 and U_2 be subspaces of a vector space V. We say that V is the *direct sum* of U_1 and U_2, notation $V = U_1 \oplus U_2$, provided that each element of V has a unique expression in the form of $v = x + y$ where $x \in U_1$ and $y \in U_2$.

(a) Show that $V = U_1 \oplus U_2$ if and only if $U_1 \cap U_2 = \{0\}$ and each element of V is expressible in the form $v = x + y$ where $x \in U_1$ and $y \in U_2$.

(b) Show that $V = U \oplus U^{\perp}$ for any subspace U of the inner product space V.

(14) Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be equipped with usual dot product and let A be an $m \times n$ matrix with real entries. Show that $\text{Ker } A = (\text{Im } A^t)^{\perp}$ and $\text{Im } A = (\text{Ker } A^t)^{\perp}$.

(15) Let A be an $n \times n$ matrix and b be a column vector in $\mathbb{R}^n$. Let $x = (x_i)$ be a column vectors of unknowns. Use the previous problem to show that only one of the following can have a solution for x

(i) $Ax = b$

(ii) $A^t x = 0$ and $x^t b \neq 0$

(This is referred as *Fredholm Alternative*)

(16) Let A be an $n \times n$ real matrix. Show that the following are equivalent

(a) A is orthogonal.

(b) A preserves length, i.e. $\|Av\| = \|v\| \ \ \forall v \in \mathbb{R}^n$.

(c) A is invertible and $A^t = A^{-1}$.

(d) The rows of A forms and orthonormal basis of $\mathbb{R}^n$.

(e) The columns of A forms an orthonormal basis of $\mathbb{R}^n$.