LA 5 & LA 6 soln.

16. Orthogonal matrix: Columns are orthonormal.

e) and a) are same.

a) $\Rightarrow$ c) From defn. $A^TA = I \Rightarrow A^T = A^{-1}$

c) $\Rightarrow$ b) $\langle v, v\rangle = \langle A^TAv, v\rangle = \langle Av, Av\rangle$ (as $\langle A^Tx, y\rangle = \langle x, Ay\rangle$)

b) $\Rightarrow$ a) $\langle A(x+y), A(x+y)\rangle = \|Ax\|^2 + \|Ax\|^2 + 2\langle Ax, Ay\rangle$.

By b) $\langle A(x+y), A(x+y)\rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$

$\Rightarrow \langle Ax, Ay\rangle = \langle x, y\rangle \;\forall\; x, y$

$\Rightarrow A^TA = I$.

c) $\Rightarrow$ e) $A^TA = I \Rightarrow AA^T = I \Rightarrow$ rows are orthonormal.

LA6

1. Since A preserves length $A(1,0) = (\cos\theta, \sin\theta)$ for some θ. Since $(1,0) \perp (0,1)$ we have $A(1,0) \perp A(0,1)$

$\Rightarrow A(0,1) = (-\sin\theta, \cos\theta)$ or $(\sin\theta, -\cos\theta)$

$\Rightarrow A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

rotation

reflexion about a line of inclination $\theta/2$.

2. Enough to choose $v = e_1$. So we need A s.t $A(e_1) = w$. Find an onb of $\mathbb{R}^n$ $\{w, w_1, \ldots, w_{n-1}\}$ by Gram-Schmidt method. $A = [w \; w_1 \cdots w_{n-1}]$, so 1st column of A is w. If $\det(A) = -1$ then interchange 2nd and 3rd column. (Pl. <u>mention</u> if A is orthogonal then $\det(A) = \pm 1$.).

3. i) $w \in N(A) \Rightarrow A(w) = 0 \Rightarrow$ for all row vectors r in A, $r \cdot w = 0$.

ii) $\dim(R(A)) + \dim(N(A)) = n$

$R(A) \oplus R(A)^\perp = \mathbb{R}^n$, $N(A) \subseteq R(A)^\perp$

$\Rightarrow N(A) = R(A)^\perp$.

4. $\det(A-\lambda I) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + a_n\lambda^n \quad \forall \lambda, \; a_n = (-1)^n$

$\Rightarrow$ by putting $\lambda = 0$, $a_0 = \det(A)$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are roots then

$$\lambda_1 \cdots \lambda_n = (-1)^n \frac{a_0}{a_n} = \det(A)$$

Expanding the determinant we see that every term is a polynomial in λ of degree at most $n-2$ except the term $(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{nn})$

$\Rightarrow$ coefficient of λ^{n-1} is $-(a_{11} + \cdots + a_{nn}) = -Tr(A)$.

5. $Av = \lambda v \Rightarrow v = \lambda A^{-1}(v) \Rightarrow A^{-1}(v) = \frac{1}{\lambda} v$

6. If λ is an eigen value of A then $(A - \alpha I)(v) = (\lambda - \alpha)v$.

If v is an eigen vector of A with eigen value u then $Av = (A - \alpha I)v + \alpha v = (u + \alpha)v$.

7. $\det(A) = \det A^t$ and $(A-B)^t = A^t - B^t$.

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $(0,1)$ is an eigen vector of A, $(1,0)$ is an eigen vector of A^t and $(1,0)$ is not an eigen vector of A ....

8. a) $Av = \lambda v \Rightarrow A^2 v = \lambda^2 v = Av = \lambda v \Rightarrow \lambda^2 v = \lambda v \Rightarrow \lambda = 0$ or 1.

b) $Av = \lambda v \Rightarrow \lambda^m v = A^m v = 0 \Rightarrow \lambda = 0$.

9. a) $(1-\lambda)^2 = 4 \Rightarrow \lambda = 3$ or $\lambda = -1$

$\lambda = 3 \rightarrow$ eigenvectors $\alpha\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$, $\alpha \in \mathbb{R}$

$\lambda = -1 \rightarrow$ " $\alpha\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$, $\alpha \in \mathbb{R}$

b) $\lambda = 0 \longrightarrow (0,-1,1)$

$\lambda = -2 \longrightarrow (-2,1,0)$

$\lambda = -3 \longrightarrow (-1,0,1)$

10. a) Eigen values are $-1, 2, 4$.

$$E_{-1} = \{(x, 0, -\tfrac{x}{2}) / x \in \mathbb{R}\} = \mathrm{sp}\{(1, 0, -\tfrac{1}{2})\}$$

$$E_2 = \{(0, y, 0) / y \in \mathbb{R}\} = \mathrm{sp}\{(0, 1, 0)\}$$

$$E_4 = \{(x, 0, 2x) / x \in \mathbb{R}\} = \mathrm{sp}\{(1, 0, 2)\}$$

$\therefore \{(1, 0, -\frac{1}{2}), (0, 1, 0), (1, 0, 2)\}$ is a basis consisting of eigen vectors.

b) Similar.

11/12. Solution will be given to the students.

---

11. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{pmatrix} \Rightarrow |A - \lambda I| = (1-\lambda)(3-\lambda)^2$

$\therefore$ Eigen values are 1 and 3.

$$E_1 = \{x : Ax = x\} \cup \{0\} = \{(x, y, z) / y = x, z = -2x, x \in \mathbb{R}\}$$
$$= \mathrm{sp}\{(1, 1, -2)\}$$

$$E_3 = \{x : Ax = 3x\} \cup \{0\} = \{(x, -x, z) / x, z \in \mathbb{R}\}$$
$$= \mathrm{sp}\{(1, -1, 0), (0, 0, 1)\}$$

$\{(1, 1, -2), (1, -1, 0), (0, 0, 1)\}$ are clearly L.I. and hence A is diagonalizable.

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \Rightarrow AQ = \begin{pmatrix} 1 & 3 & 0 \\ 1 & -3 & 0 \\ -2 & 0 & 3 \end{pmatrix} = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow Q^{-1} A Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

12. $A = \begin{pmatrix} 7 & -5 & 15 \\ 6 & -4 & 15 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow |A - \lambda I| = (\lambda-1)^2(\lambda-2) \Rightarrow \lambda = 1, \lambda = 2.$

$E_1 = \{(x,y,z) / 6x - 5y + 15z = 0\}$

$= \text{span}\,\{(1, 0, -\frac{6}{15}), (0, 1, 1/3)\}$

$E_2 = \{(x,y,z) / \begin{pmatrix} 5 & -5 & 15 \\ 6 & -6 & 15 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\} \cup \{(0,0,0)\}$

$= \{(x, x, 0) / x \in \mathbb{R}\} = \text{Span}\,\{(1,1,0)\}.$

$\therefore \{(1,0,-\frac{6}{15}), (0,1,1/3), (1,1,0)\}$ is LI & hence a basis consisting of eigen vectors.

Let $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -6/15 & 1/3 & 0 \end{pmatrix}$. Then $AQ = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -6/15 & 1/3 & 0 \end{pmatrix} = Q\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

As $AQ = QD \Rightarrow A = QDQ^{-1}$ we get

$A^6 = (QDQ^{-1})^6 = (QDQ^{-1})(QDQ^{-1})(QDQ^{-1})(QDQ^{-1})(QDQ^{-1})(QDQ^{-1})$

$= QD^6Q^{-1}$

$= Q\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^6 \end{pmatrix}Q^{-1}$

—