MTH102N
ASSIGNMENT–C2

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(1) Find polar representation of all complex numbers satisfying $z^5 = -4$.

(2) Show that the function $f(z) = f(x+iy) = \sqrt{|xy|}$ satisfies Cauchy Riemann Equations at 0 but it is not differentiable at 0.

(3) Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there do not exist any point c on the line $y = 1 - x$ joining z_1 and z_2 such that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$$

(Mean value theorem does not extend to complex derivatives).

(4) Suppose that $g : \mathbb{D} \to \mathbb{C}$ is an analytic function with zero derivative. Prove that g is a constant function.

Suppose that $g : \mathbb{D} \cup \{z : |z-3| < 1\} \to \mathbb{C}$ is an analytic function with zero derivative. Is g necessaryly a constant function?

(5) If f is a differentiable function in an open set Ω, prove that

$$\frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) = 0, \text{ and } f' = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}).$$

(6) Let U be an open set and $f\colon U \to \mathbb{C}$ be a differentiable function. Let $\overline{U} := \{\overline{z}\colon z \in U\}$. Show that the function $g\colon \overline{U} \to \mathbb{C}$ defined by $g(z) := \overline{f(\overline{z})}$ is differentiable on $\overline{U}$.

(7) Let Ω be an open connected subset of $\mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ be a differentiable function. Show that the function $f = u + iv$ is constant if
   (a) either of the functions u or v is constant, or
   (b) $|f(z)|$ is constant for all $z \in \Omega$, or
   (c) if there exists an $\alpha \in \mathbb{R}$ such that $f(z) = |f(z)|e^{i\alpha}$ for all $z \in \Omega$.

(8) Let $f : \mathbb{D} \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}$, f(z)=f(w) whenever $|z| = |w|$. Prove that f is a constant function. (Use CR equations in polar coordinates)

(9) Let $f = u + iv$ is an analytic function defined on the whole of $\mathbb{C}$. If $u(x,y) = \phi(x)$ and $v(x,y) = \psi(y)$ prove that, for all $z \in \mathbb{C}$, $f(z) = az + b$ for some $a \in \mathbb{C}$, $b \in \mathbb{C}$.