MTH102N
ASSIGNMENT–C3

(1) Determine *all* $z \in \mathbb{C}$ for which the following series are absolutely convergent.

$$a)\ \sum \frac{z^n}{n^2},\quad b)\ \sum \frac{z^n}{n!},\quad c)\ \sum \frac{1}{n!}(\frac{1}{z})^n,\quad d)\ \sum \frac{1}{2^n}\frac{1}{z^n}.$$

(2) Let $a_n = \dfrac{(-1)^n}{\sqrt{n}} + i\dfrac{1}{n^2}$ for $n = 1, 2, 3 \ldots\ldots$. Show that the series $\sum_n^\infty a_n$ converges but it does not converge absolutely.

(3) The following series $\sum_{n=0}^\infty z^n$, $\sum_{n=0}^\infty \dfrac{z^n}{n}$ and $\sum_{n=0}^\infty \dfrac{z^n}{n^2}$ have radius of convergence 1. Show that the series
  (a) $\sum_n z^n$ does not converge for any z such that $|z| = 1$,
  (b) $\sum_{n=0}^\infty z^n/n$ converges for all $z \neq 1$ such that $|z| = 1$ and
  (c) $\sum_{n=0}^\infty z^n/n^2$ converges for all z such that $|z| = 1$.

(4) Find the radius of convergence of $a)$ $\sum_n 2nz^{2n}$, $b)$ $\sum_n n!z^{2n+1}$, $c)$ $\sum_n (-1)^n \frac{z^{2n}}{2n!}$.

(5) Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus \{m\}$, for all $m \in \mathbb{N} \cup \{0\}$). Prove that the radius of convergence of the series

$$\sum_{n=0}^\infty \frac{\alpha(\alpha+1)\ldots(\alpha+n)}{\beta(\beta+1)\ldots(\beta+n)} z^n$$

is either 1 or infinity.
(Hint. What is the *series* if $\alpha = -m$, for some $m \in \mathbb{N} \cup \{0\}$)?)

(6) Let α, $\beta \in \mathbb{C}$ be such that $|\alpha| < |\beta|$. Find the radius of convergence of the power series $\sum_{n=0}^\infty (3\alpha^n - 5\beta^n)z^n$.

(7) Let R_1 and R_2 be the radii of convergence of the series $\sum_n a_n z^n$ and $\sum_n b_n x^n$ respectively. Show that the radius of convergence R of the series $\sum_n (a_n + b_n)z^n$ satisfies $R \geq \min\{R_1, R_2\}$.

(8) If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = R$ show that the radius of convergence of $\sum_{n=1}^\infty n a_n z^{n-1}$ is R.

(9) Show that $\sum_{n=0}^\infty (n+1)^2 z^n = (1+z)/(1-z)^3$, $|z| < 1$.

(10) Find i^i and $\cosh(\text{Log } 4)$
(Log stands for the principal branch of the logarithm).

(11) For $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ is it always true that $\text{Log }(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2$? Find the conditions on z_1, z_2 so that the equality holds.