MTH102N
ASSIGNMENT–C4

(1) Evaluate the integral $\int_\Gamma ze^{z^2}dz$ where Γ is the curve from 0 to $1+i$ taken along the parabola $y = x^2$.

(2) Let $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and f be an analytic function defined on $\mathbb{D}$. Suppose $a, b \in \mathbb{D}$ and $\gamma(t) = a + t(b-a)$, $t \in [0,1]$ is the straight line joining a and b.

(a) Prove that $\frac{f(b)-f(a)}{b-a} = \int_0^1 f'(\gamma(t)dt$.

(b) If Re $f'(z) > 0$ for all $z \in \mathbb{D}$ then prove that f is injective.

(3) Show that $\displaystyle\int_\gamma \frac{e^{az}}{z^2+1} = 2\pi i \sin a$, if $\gamma(\theta) = 2e^{i\theta}$ for $0 \leq \theta \leq 2\pi$.

(4) Evaluate the following integrals:

$$a) \int_0^{2\pi} e^{e^{i\theta}} d\theta, \qquad b) \int_0^{2\pi} e^{e^{i\theta}-i\theta} d\theta.$$

(5) Let C denotes the unit circle with counterclockwise orientation.

(a) Evaluate the integral

$$\int_C \left(\frac{z-2}{2z-1}\right)^3 dz.$$

(b) Without using Cauchy's integral formula evaluate $\int_C \frac{\sin z}{z} dz$.

(6) Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\{z \in \mathbb{C} : z \neq 0\}$ and bounded on the set $\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. Prove that $\int_{|z|=R} f(z)dz = 0$ for every $R > 0$.

(7) (Mean Value theorem) Let Ω be a simply connected domain and $f : \Omega \to \mathbb{C}$ be an analytic function. Then prove that $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$ for every $r > 0$ such that $B(z_0, r)$ is contained in Ω.

(8) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leq A + B|z|^k$ for some $k \in \mathbb{N}$ and for some positive real numbers A and B. Show that f is a polynomial of degree at most k.

(9) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\displaystyle\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$. Show that f is constant.

(10) Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the image of the function has to necessarily meet the real axis and imaginary axis.

(11) Let $f : D \to D$ be an analytic function such that $f(0) = 0$. Show that
   (a) $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$ and $|f'(0)| \leq 1$.
   (b) If $|f(z_0)| = |z_0|$ for some $z_0 \in D$ or $|f'(0)| = 1$, then there exist $c \in \mathbb{C}$ such that $|c| = 1$ and $f(z) = cz$ for all $z \in D$.

(12) Let $f_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$ be analytic functions such that $f_1(a_n) = f_2(a_n)$ for a bounded sequence of complex numbers. Show that the functions are same.

(13) Find the maximum of the function $|f|$ if
   (a) $f(z) = z^2 - z$ on $\overline{D}$.
   (b) $f(z) = \sin z$ on $\overline{D}$.