MTH102N
ASSIGNMENT–C6

Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$ *should be interpreted as* $\lim_{R\to\infty} \int_{-R}^{R} f(x)dx$.

(1) Suppose f and g are analytic functions in a neighbourhood of a point $z_0 \in \mathbb{C}$ such that $g(z_0) \neq 0$ and f has a simple zero at z_0. Prove that

$$\mathrm{Res}(\frac{g}{f} : z_0) = \frac{g(z_0)}{f'(z_0)}.$$

(2) Let f be analytic in a domain Ω and γ be a simple closed curve in Ω in the counterclockwise sense. Suppose z_0 is the only zero of f in the region enclosed by Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i m,$$

where m is the order of zero of f at z_0.

(3) Find the isolated singularities and compute the residue of the functions

$$a)\ \frac{e^z}{z^2-1}, \quad b)\ \frac{3z}{z^2+iz+2}, \quad c)\ \cot \pi z.$$

(4) let $f(z) = \dfrac{\pi \cot \pi z}{(z+\frac{1}{2})^2}$. Compute the residue of f at isolated singularities.

(5) Prove Jordan's inequality: Given any $R > 0$, $\displaystyle\int_0^\pi e^{-R\sin\theta} d\theta < \frac{\pi}{R}$.

(6) Evaluate:

$$(a)\ \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^n} dx,\ n \geq 1,\ (b)\ \int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2+a^2} dx,\ a > 0.$$

(Hint: For b) use Jordan's inequality.)

(7) Prove that $\displaystyle\int_0^\pi \sin^{2n}\theta d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi$.

(8) Compute the following integrals:

$$(a)\ \int_{-\infty}^{\infty} \frac{\sin x}{x} dx,\ (b)\ \int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx,\ (c)\ \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1} dx,\ \text{for } 0 < a < 1.$$

(9) Show that $\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}$, for $\xi \in \mathbb{R}$ by integrating the differentiable function $f(z) = e^{-z^2}$ along the lines of the rectangle with vertices $R,\ R+i\xi,\ -R+i\xi,\ -R$.

(10) Given $a > 0$, prove that $\int_{\infty}^{\infty} \frac{\cos ax}{1+x^2} dx = \pi e^{-a}$.

(11) Find the image of the right half plane under the mapping $\phi(z) = i\frac{1-z}{1+z}$.

(12) What is the image of the strip $S = \{z = x + iy : 0 < y < 2\}$ under the Mobius transformation $\phi(z) = \frac{z}{z-i}$?