Lecture 1

Many problems in Science & Engineering can be converted into problems involving matrices and they can be solved using matrix theory. In this course we will be studying about matrices. Let us start with the system of linear equations in which the matrices occur naturally. We start with some examples.

Example 1:

$$\left.\begin{aligned} x+y&=1 \\ 3x+2y&=2 \end{aligned}\right\}$$

Since the equations represent lines, they either: (a) intersect at a point (b) are parallel (c) represent the same line. Depending on the situation we will get: (a) one solution (b) no solution (c) infinite number of solutions. To find the solutions, we use the elementary elimination method: the equations can be transformed to

$$\left.\begin{aligned} x+y&=1 \\ 3x+2y&=2 \end{aligned}\right\} \xrightarrow{2R_1} \left.\begin{aligned} 2x+2y&=2 \\ 3x+2y&=2 \end{aligned}\right\} \xrightarrow{R_2-R_1} \left.\begin{aligned} 2x+2y&=2 \\ x&=0 \end{aligned}\right\} \longrightarrow \begin{aligned} y&=1 \\ x&=0 \end{aligned}$$

Example 2:

$$\left.\begin{aligned} 2x+y+z&=5 \\ 4x-6y&=-2 \\ -2x+7y+2z&=9 \end{aligned}\right\} \xrightarrow[R_3+R_1]{R_2-2R_1} \left.\begin{aligned} 2x+y+z&=5 \\ -8y-2z&=-12 \\ 8y+3z&=14 \end{aligned}\right\} \xrightarrow{R_3+R_2} \begin{aligned} 2x+y+z&=5 \\ -8y-2z&=-12 \\ z&=2. \end{aligned}$$

If we are given a larger system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

how do we conclude whether the system has a solution and then compute it. To tackle, we analyse the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Before proceeding, we recall some elementary fact about matrices operations.

1. Addition: Two $m \times n$ matrices are added the following way:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Symbolically: $(a_{ij})_{m\times n} + (b_{ij})_{m \times n} = (a_{ij}+b_{ij})_{m\times n}$.

2. Multiplication: An $m\times n$ matrix is multiplied to an $n \times l$ matrix to produce $m \times l$ matrix by the following rule:

$$(a_{ij})_{m\times n} \, (b_{ij})_{n\times l} = (c_{ij})_{m\times l} \quad \text{where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

3. Scalar multiplication: If $\alpha \in \mathbb{R}$, then $\alpha\,(a_{ij})_{m\times n} = (\alpha\, a_{ij})_{m\times n}$.

Remark: It can happen that $AB \neq BA$ for some matrix A & B.

For example $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Note that in terms of matrices our system of linear equations can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{or} \quad Ax = b$$

where A is $m\times n$; x is $n \times 1$ and b is $m \times 1$ matrices.

The advantage of doing this can be seen in the next result.

Proposition: Every system of linear equations of the form $Ax = b$ has either no solution, one solution or infinitely many solutions. (so having two solutions means that there are infinitely many solutions).

Proof: Suppose there exists two solutions u & v s.t $Au = b$ and $Av = b$ & $u \neq v$. Then $A(u-v) = 0 \Rightarrow A(\alpha(u-v)) = 0 \;\; \forall \alpha \in \mathbb{R}$

$\Rightarrow u + \alpha(u-v)$ is a solution $\forall \alpha \in \mathbb{R}$. ∎

Cor: If a homogeneous system $Ax = 0$ has a non zero solution $\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$ (ie one $u_i \neq 0$) then it has infinitely many solutions.

Now what about finding the solutions of the system of equations.
For that, let us analyse what we did in the first example:
we performed the following operations for solving:

$$\left.\begin{array}{l} x+y=1 \\ 3x+2y=2 \end{array}\right\} \xrightarrow{2R_1} \left.\begin{array}{l} 2x+2y=2 \\ 3x+2y=2 \end{array}\right\} \xrightarrow{R_2-R_1} \begin{array}{l} 2x+2y=2 \\ x \quad\;\; =0. \end{array}$$

We will view the above operations in a different way.
Consider the augmented matrix of the original system of equations:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & 2 \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & 2 \end{array}\right).$$

If we perform the operation $2R_1$ on the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
we get the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Note that

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & 2 \end{array}\right) = \left(\begin{array}{cc|c} 2 & 2 & 2 \\ 3 & 2 & 2 \end{array}\right)$$

which is the augmented matrix of the system of equations obtained after performing the operation $2R_1$ on the original system of equations. So the operation $2R_1$ on the system of equations is equivalent to (pre) multiplying the augmented matrix by a matrix obtained by executing the same operation on the identity matrix.

In a similar way, let us now illustrate the second operation R_2-R_1 performed in the previous example. If we perform the operation R_2-R_1 on $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we get $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Note that performing R_2-R_1 on the system of equation is equivalent to:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 & 2 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

With respect to the original augmented matrix, we have done:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$

to obtain the augmented matrix of the equations after performing $2R_1$ & R_2-R_1.

The matrices $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ & $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ are called elementary matrices which will be defined formally in the next lecture.