Lecture10

As we mentioned earlier we will elaborate the study of matrices in the framework of linear spaces. We will basically view a matrix as a map from a linear space U to a linear space V. We will see how these additional structures on U & V play a role.

Linear transformation (or linear map): Let U and V be two linear spaces. A map $T: U \to V$ is called a linear map if

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$ for $u_1, u_2 \in U$

(ii) $T(\alpha u) = \alpha T(u)$, $\alpha \in \mathbb{R}$ and $u \in U$.

From conditions (i) & (ii) we see that linear maps preserve algebraic operations. Note that T satisfies (i) & (ii) iff

$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$ for $\alpha, \beta \in \mathbb{R}$ and $u_1, u_2 \in U$.

We will see that a matrix is a linear map.

Examples:

1. $T: \mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = (x, 3x)$, $x \in \mathbb{R}$ is a linear map.
2. $T_i: \mathbb{R}^n \to \mathbb{R}$ defined by $T_i(x_1, x_2, \ldots, x_n) = x_i$ is a linear map
3. For an $m\times n$ matrix, define $T_A: \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(x) = Ax$. Then T_A is linear. (Here Ax means Ax^t).
4. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (xy, xy)$ is not linear
5. $T: \mathbb{R}^{n+1} \to \mathbb{P}_n$ defined by $T(a_0, a_1, \cdots a_n) = a_0 + a_1 x + \cdots + a_n x^n$ is linear
6. $\frac{d}{dx}: \mathbb{P}_n \to \mathbb{P}_n$, $\frac{d}{dx}(P_n(x)) = P_n'(x)$ is linear
7. $T: \mathbb{P}_n \to \mathbb{R}$, $T(P_n(x)) = \int_0^1 P_n(x)$ is linear.

Proposition: If T is a linear map then $T(0)=0$, $T(-u) = -T(u)$ & $T(u_1-u_2) = T(u_1)-T(u_2)$.

Proof: Exercise.

Remark: $T \equiv 0$ is a linear map.

Kernal and Range of a linear map: Let $T: V \to W$ be a linear map. Then the kernal of T, denoted by $N(T)$, is defined by

$N(T) = \{v \in V : Tv = 0\}$ and the range of T, denoted by $R(T)$ is defined by $R(T) = T(V) = \{w \in W : \exists\, v \in V \text{ s.t. } Tv = w\}$.

Exercise: Verify that $N(T)$ & $R(T)$ are subspaces of V & W resp.

Example: Find the range & kernel of $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x,y,z) = (x+z,\ x+y+2z,\ 2x+y+3z).$$

Note that
$$\begin{aligned} N(T) &= \{(x,y,z) \in \mathbb{R}^3 : T(x,y,z) = (0,0,0)\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : x+z=0,\ x+y+2z=0\ \&\ 2x+y+3z=0\} \\ &= \{(x,x,-x) : x \in \mathbb{R}\} = \operatorname{span}\{(1,1,-1)\}. \end{aligned}$$

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ & $e_3 = (0,0,1)$. Then

$$\begin{aligned} R(T) = \{Tu : u \in \mathbb{R}^3\} &= \{T(\alpha e_1 + \beta e_2 + \gamma e_3) : \alpha, \beta, \gamma \in \mathbb{R}\} \\ &= \{\alpha Te_1 + \beta Te_2 + \gamma Te_3 : \alpha, \beta, \gamma \in \mathbb{R}\} \\ &= \operatorname{span}\{Te_1, Te_2, Te_3\} \\ &= \operatorname{span}\{(1,1,2), (0,1,1), (1,2,3)\} \\ &= \operatorname{span}\{(1,1,2), (0,1,1)\} \end{aligned}$$

Rank & Nullity: Rank of $T = \dim(R(T))$ & Nullity of $T = \dim N(T)$.

Remark: In the previous example, we have seen that

$$\text{Rank}(T) + \text{Nullity}(T) = 3 = \dim(\mathbb{R}^3).$$

The relation between Rank & Nullity is given in the following theorem.

Rank-Nullity Theorem: Let V & W be f.d.v.s. & $T : V \to W$ be a linear map. Then $\text{Rank}(T) + \text{Nullity}(T) = \dim(V)$.

Proof: Let $\{u_1, u_2, \ldots, u_k\}$ be a basis of $N(T)$. Extend this set to a basis of V, say $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_n\}$.

<u>claim:</u> $\{TV_1, \ldots, TV_n\}$ is a basis of $R(T)$.

It is easy to verify that $\text{span}\{TV_1, \ldots, TV_n\} = R(T)$.

To show the set is L.I., let $\sum_1^n \alpha_i TV_i = 0$. Then

$$T\left(\sum_1^n \alpha_i V_i\right) = 0 \Rightarrow \sum_1^n \alpha_i V_i \in N(T)$$

$$\Rightarrow \sum_1^n \alpha_i V_i = \sum_1^k \beta_i u_i \text{ for some } \beta_i$$

$$\Rightarrow \sum_1^n \alpha_i V_i - \sum_1^k \beta_i u_i = 0$$

$$\Rightarrow \alpha_i = 0 \ \& \ \beta_j = 0, \ i = 1, \ldots, n \ \& \ j = 1, 2, \ldots, k$$

(because $\{u_1, \ldots, u_k, v_1, \ldots, v_n\}$ is L I) ∎

<u>Applications:</u> 1. Let V & W be a f.d.v.s of dimension n & $T: V \to W$ be a linear map. Then T is one-one iff T is onto.

<u>Proof:</u> T is one-one $\Leftrightarrow N(T) = \{0\} \Leftrightarrow \text{Nullity}(T) = 0 \Leftrightarrow$
$\text{Rank}(T) = n \Leftrightarrow R(T) = V \Leftrightarrow T$ is onto

2. Let V be a f.d.v.s & $T: V \to V$ be linear. If T is one-one or onto then T is invertible.

(*) <u>Rank of a matrix:</u> Let $A = (a_{ij})$ be an $m \times n$ matrix. Suppose $A = (C_1 \ C_2 \ldots C_n)$ where C_i's are columns of A. Then $A: \mathbb{R}^n \to \mathbb{R}^m$ and $\text{Rank}(A) = \dim(R(A)) \leq m$.

Note that $R(A) = \text{span}\{Ae_1, Ae_2, \ldots, Ae_n\} = \text{Span}\{C_1, \ldots, C_n\}$; this space is sometimes called column space of A because it is spanned by column vectors of A.

Similarly a row space of A is defined as follows:

Row space of A = $\text{span}\{R_1, \ldots, R_m\}$, span of row vectors of A.

(*) 3. Let A be an $m \times n$ matrix. Then $Ax = 0$ has a nontrivial solution if the number of unknowns are more than the number of equations; because $A: \mathbb{R}^n \to \mathbb{R}^m$, $m < n$ & $\text{nullity}(A) \neq \{0\}$