Row-rank & column rank: The dim(Row space) & dim(column space) are called row rank & column rank respectively.

Theorem: The row rank & column rank of any $m\times n$ matrix are equal.

Proof(*): Let $A=(a_{ij})_{n\times m}$. Suppose row rank $=k$ & $\{v_1, v_2, \dots v_k\}$ be a basis of the row space. Let $v_r=(b_{r1}, b_{r2}, \dots, b_{rn})$, for $r=1,2,\dots,k$. Since $\{v_1,\dots,v_k\}$ is a basis for the row space, each

$$R_i=(a_{i1}, a_{i2},\dots,a_{in})=\sum_{r=1}^{k}\alpha_{ir}v_r=\sum_{r=1}^{k}\alpha_{ir}(b_{r1}, b_{r2},\dots,b_{rn}) \text{ for some } \alpha_{ir}.$$

ie $$(a_{i1}, a_{i2},\dots,a_{in})=\left(\sum_{r=1}^{k}\alpha_{ir}b_{r1}, \sum_{r=1}^{k}\alpha_{ir}b_{r2},\dots,\sum_{r=1}^{k}\alpha_{ir}b_{rn}\right),$$ for $1\le i\le m$.

To compare k with the column rank, let us look at the columns of A:

$$a_{1j}=\sum_{r=1}^{k}\alpha_{1r}b_{rj}=\alpha_{11}b_{1j}+\alpha_{12}b_{2j}+\dots+\alpha_{1k}b_{kj}$$

$$a_{2j}=\sum_{r=1}^{k}\alpha_{2r}b_{rj}=\alpha_{21}b_{1j}+\alpha_{22}b_{2j}+\dots+\alpha_{2k}b_{kj}$$

$$\vdots$$

$$a_{mj}=\sum_{r=1}^{k}\alpha_{mr}b_{rj}=\alpha_{m1}b_{1j}+\alpha_{m2}b_{2j}+\dots+\alpha_{mk}b_{kj}.$$

This implies that
$$C_j=j\text{th column of } A=\begin{pmatrix}a_{1j}\\a_{2j}\\\vdots\\a_{mj}\end{pmatrix}=b_{1j}\begin{pmatrix}\alpha_{11}\\\alpha_{12}\\\vdots\\\alpha_{m1}\end{pmatrix}+\dots+b_{kj}\begin{pmatrix}\alpha_{1k}\\\alpha_{2k}\\\vdots\\\alpha_{mk}\end{pmatrix}.$$

$\Rightarrow$ C_j is a linear combination of k vectors.

$\Rightarrow$ column rank $\le$ row rank.

A similar argument will give the reverse inequality ∎

Definition: For an $m\times n$ matrix A, the rank of A is defined to be the row rank or column rank of A and it is denoted by rank A.

<u>Example:</u> Determine the rank of A, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

<u>Solution:</u> By row operations, we get:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the row rank = column rank = rank A = 2.

Note that one can also do a sequence of column operations and find the rank.

<u>Solvability of a system (Application):</u>

In one of the previous lectures, we considered the solvability of a system of n equations with n unknowns. We will now consider a system of m linear equations with n unknowns.

Let A be an $m \times n$ matrix. Consider the system $Ax = b$. If we consider A as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ i.e. $A: \mathbb{R}^n \to \mathbb{R}^m$, as a consequence of Rank-Nullity theorem, we get the following results:

1. <u>Existence:</u>

$Ax = b$ has a solution $\iff b \in R(A)$ - range of A

$\iff b \in \text{span}\{\text{column vectors of } A\}$.
i.e. column space

$\iff$ A & (A, b) have the same rank
- (A, b) is the Augmented matrix.

2. <u>Uniqueness:</u>

Suppose $Ax = b$ has a solution. Then the solution is unique $\iff$ $Ax = 0$ has only the trivial solution $x = 0$.

$\iff \ker(A) = N(A) = 0$

$\iff$ Nullity$(A) = 0$

$\iff$ rank$(A) = n$.

(Note that here n could be less than m).

<u>Theorem:</u> Consider the system of equations $Ax = b$. Let the rank of A be r & A be an $m \times n$ matrix (so $A: \mathbb{R}^n \to \mathbb{R}^m$). Then

(1). $Ax=b$ has a solution $\iff$ (A,b) has rank r.

(2). If $r=m$ then $Ax=b$ has a solution for every $b\in\mathbb{R}^m$.

(3). If $r=m=n$, then $Ax=b$ has a unique solution for every $b\in\mathbb{R}^m$.

(4). If $r=m<n$, then for every $b\in\mathbb{R}^m$, $Ax=b$ has infinite no. of solutions.

(5). If (i) $r<m=n$ (ii) $r<m<n$ (iii) $r<n<m$ and if $Ax=b$ has a solution, then it has infinite number of solutions.

(6). If $r=n<m$ and if $Ax=b$ has a solution, then the solution is unique.

<u>Proof</u>: We have already seen the proofs of (1), (2), (3) and (6) above.

Let us see the proofs of (4) & (5)

<u>Proof of (4)</u>: First note that, since $r=m$, the range of A is $\mathbb{R}^m$. Therefore, for every $b\in\mathbb{R}^m$, there is a solution for $Ax=b$. By Rank-Nullity Theorem, the $\dim(N(A)) = n-r>0$. Therefore $N(A)$ has infinite number of solutions and hence, $Ax=b$ has infinite number of solutions.

<u>Proof of (5)</u>: The proof is similar to the proof of (4). If $Ax_0=b$ for some x_0, then every x_0+x, $x\in N(A)$ is a solution.

<u>Example</u>: Consider the system

$$2x_1 + x_3 - x_4 + x_5 = 2$$
$$x_1 + x_3 - x_4 + x_5 = 1$$
$$12x_1 + 2x_2 + 8x_3 + 2x_5 = 12.$$

The augmented matrix (A,b) is

$$\begin{bmatrix} 2 & 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & -1 & 1 & 1 \\ 12 & 2 & 8 & 0 & 2 & 12 \end{bmatrix} \xrightarrow[\text{operations}]{\text{After few}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$ $\text{Rank}(A) = \text{Rank}(A,b) = 3$. (Here $A:\mathbb{R}^5\to\mathbb{R}^3$).

Therefore, by Rank-Nullity Theorem,

$$\dim(N(A)) = \dim(\text{dom}\,A) - \dim(\text{Range}) = 5-3=2.$$

So the set of solutions of the system is the translation of $N(A)$.

Let us find the set of solutions. Note that

$$\left.\begin{aligned} x_1 &= 1 \\ x_2 + 4x_4 - 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} x_1 &= 1 \\ x_2 &= -4x_4 + 3x_5 \\ x_3 &= x_4 - x_5 \end{aligned}$$

If we choose x_4 & x_5 arbitrarily, the set of solutions can be written as

$$\{(1, -4x_4 + 3x_5, x_4 - x_5, x_4, x_5) : x_4, x_5 \in \mathbb{R}\}$$

$$= \{(1,0,0,0,0) + x_4(0,-4,1,1,0) + x_5(0,3,-1,0,1) : x_4, x_5 \in \mathbb{R}\}.$$

$$= (1,0,0,0,0) + \text{span}\{(0,-4,1,1,0), (0,3,-1,0,1)\}$$

$$= (1,0,0,0,0) + N(A). \blacksquare$$

__Example 2:__ Consider the system:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 4 \\ 2x_1 - x_3 - 3x_4 &= 4 \\ x_1 - 2x_2 - x_3 &= 0 \\ 3x_1 + x_2 - x_3 - 5x_4 &= 5 \end{aligned}$$

The augmented matrix is,

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 4 \\ 2 & 0 & -1 & -3 & 4 \\ 1 & -2 & -1 & 0 & 0 \\ 3 & 1 & -1 & -5 & 5 \end{bmatrix} \xrightarrow[\text{iterations}]{\text{After few}} \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{7} \end{bmatrix}.$$

Here $\text{Rank}(A) = 3 < \text{Rank}(A, b) = 4$. So the system can't have a solution. In other words, we say that system is inconsistant. $\blacksquare$