Lecture 12

Some applications: 1. Lagrange interpolation:

Let $D = \{(x_i, y_i) : i = 0, 1, 2, \dots, n\} \subseteq \mathbb{R}^2$. We want to find a polynomial $p(x)$ of degree less than or equal to n such that

$$p(x_i) = y_i, \quad i = 0, 1, 2, \dots, n$$

To find such $p(x)$, let $p(x) = a_0 + a_1 x + \dots + a_n x^n$, where a_i's are unknowns. Since $p(x_i) = y_i$, $i = 1, 2, \dots, n$, we have

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Since the determinant of the above matrix is non-zero, the above system has a unique solution. Such $p(x)$ is called interpolating polynomial.

2. The Wronskian: Let $V = C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}$. Then $(V, +, \cdot)$ with the usual addition $+$ of two functions and the usual scalar multiplication $\cdot$, V is a vector space. Many times we would like to know whether given a finite subset of V is L.I or L.D. For example, you will come across this problem in the differential equation course.

For example, consider the set of functions $\{x, \sin x, \cos x\}$. To verify whether this set is L.I, let

$$\alpha_1 x + \alpha_2 \sin x + \alpha_3 \cos x = 0$$

Note that here 0 means 0 function. If the above equation implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then the set is L.I. One way is that take different values of x. For example if we take $x = 0$, then $\alpha_3 = 0$. Then take $x = 2\pi$ to show that $\alpha_1 = 0$. Then take $x = \pi/2$ to show that $\alpha_2 = 0$. But this method will not work in general. Instead from the above equation, by differentiating the LH & RHS, we get

$$\left.\begin{aligned} \alpha_1 x + \alpha_2 \sin x + \alpha_3 \cos x &= 0 \\ \alpha_1 + \alpha_2 \cos x - \alpha_3 \sin x &= 0 \\ -\alpha_3 \sin x + \alpha_3 \cos x &= 0 \end{aligned}\right\} \Longrightarrow \begin{bmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that the determinant of the matrix $W(x) = x$, which is nonzero for $x \neq 0$. Hence $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$ is the only solution for the above system. Hence the set is L.I. This method can be generalized to any finite subset of $C(\mathbb{R})$. The determinant $W(x)$ is called the Wronskian of the given set.

Exercise: Show that the set $\{x, e^x, e^{-x}\}$ is L.I.

Inner product spaces:

We have seen that a vector space $(V, +, \cdot)$ is a generalization of the structure $(\mathbb{R}^3, +, \cdot)$. In V, one can view dimension one subspace of V as a line and two as a plane geometrically. But we know that geometrically $\mathbb{R}^3$ is rich because, we have additional structures such as angle and distance between two vectors. We will now generalize these additional structures to arbitrary linear spaces.

Consider the scalar product in $\mathbb{R}^3$: $X_1 = (x_1, y_1, z_1)$, $X_2 = (x_2, y_2, z_2)$, then $X_1 \cdot X_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$. We know that this scalar product satisfies the conditions:

$X \cdot (Y + \alpha Z) = X \cdot Y + \alpha\, Y \cdot Z,\ \alpha \in \mathbb{R}$, $X \cdot Y = Y \cdot X$, $X \cdot X \geq 0 \ldots\ldots$

We now generalize the scalar product to arbitrary linear spaces.

Definition: Let V be a real vector space. An inner product, denoted by $\langle \cdot, \cdot \rangle$, is a map from $V \times V$ to $\mathbb{R}$ ie $\langle , \rangle : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties: For $x, y, z \in V$ & $\alpha \in \mathbb{R}$,

(i) $\langle x, x \rangle \geq 0$ & $\langle x, x \rangle = 0 \iff x = 0$

(ii) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(iii) $\langle x, y \rangle = \langle y, x \rangle$

(iv) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

The space $(V, \langle , \rangle)$, ie $(V, +, \cdot, \langle , \rangle)$ is called an inner product space.

Examples: 1. The dot product or the scalar product in $\mathbb{R}^n$ is an inner product (i.p) in $\mathbb{R}^n$.

2. For $x = (x_1, x_2)$ & $y = (y_1, y_2) \in \mathbb{R}^2$ define $\langle x, y \rangle = y_1(x_1 + 2x_2) + y_2(2x_1 + 5x_2)$. Then $\langle , \rangle$ is an i.p. in $\mathbb{R}^2$. To verify $\langle x, x \rangle = 0 \Rightarrow x = 0$, note that $\langle x, x \rangle = (x_1 + 2x_2)^2 + x_2^2$.

3. Let $V = C[0,1]$ be the vector space of all real valued continuous functions on $[0,1]$. For $f, g \in C[0,1]$, define $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$. Verify that $\langle , \rangle$ is an inner product.

Length: For $u \in (V, \langle \rangle)$, define $\|u\| = \sqrt{\langle u, u \rangle}$ - the distance between 0 and u.

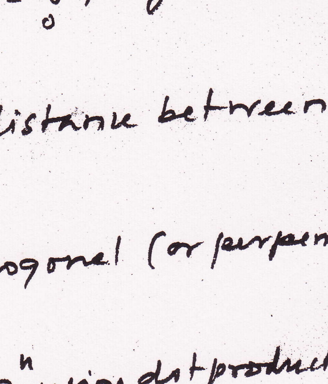
orthogonal: Two vectors u & $v \in V$ are said to be orthogonal (or perpendicular) if $\langle u, v \rangle = 0$.

These definitions are similar to what we do in $\mathbb{R}^n$ using dot product.

Theorem: (Cauchy-Schwarz inequality). Let $u, v \in (V, \langle \rangle)$. Then

$|\langle u, v \rangle| \leq \|u\| \, \|v\|$.

Proof: The proof is geometric in nature. So just imagine that we are in $\mathbb{R}^2$. If $u = 0$, the inequality is true. Suppose $u \neq 0$ & $\|u\| = 1$. So we have to show that $|\langle u, v \rangle| \leq \|v\|$.



Note that $\|v - \langle u, v \rangle u\|^2 = \langle v - \langle u, v \rangle u, v - \langle u, v \rangle u \rangle$

$= \|v\|^2 - |\langle u, v \rangle|^2$

(This is expected because in $\mathbb{R}^2$ this is nothing but Pythagoras theorem)

$\Rightarrow |\langle u, v \rangle|^2 \leq \|v\|^2 \Rightarrow |\langle u, v \rangle| \leq \|v\|$

If $\|u\| \neq 1$, consider $u/\|u\|$ instead of u.