Lecture 13

In the previous lecture we have seen the Cauchy-Schwarz inequality : $|\langle u, v\rangle| \leq \|u\|\|v\|$.

Problem: Show that $\|u+v\| \leq \|u\| + \|v\|$.

Definition: A set of vectors is said to be orthogonal if each pair of distinct vectors of the set is orthogonal.

Proposition Any orthogonal set of non-zero vectors in an i.p.s is L.I.

Proof: Let A be an orthogonal set, $u_1, u_2, \ldots, u_n \in A$ and

$$\alpha_1 u_1 + \cdots + \alpha_n u_n = 0.$$

Then $\langle \alpha_1 u_1 + \cdots + \alpha_n u_n, u_i\rangle = \alpha_i \langle u_i, u_i\rangle = \alpha_i \|u_i\|^2 = 0 \Rightarrow \alpha_i = 0.$ ∎

The converse of the above result is not true.

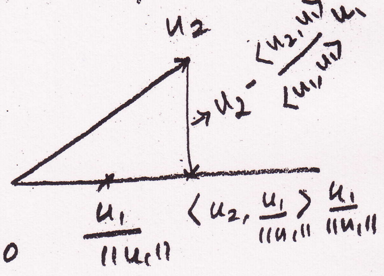
Examples: 1. $\{(1,1),(1,0)\}$ is L.I but not an orthogonal set.

2. $\{(1,1),(1,-1)\}$ - an orthogonal set

3. The set $\{1, x, x^2\}$ is L.I in $\mathbb{P}_2[0,1]$ but not orthogonal w.r.to the usual inner product as $\langle 1, x\rangle = \int_0^1 1.x\,dx = \frac{1}{2}$.

Theorem: Every f.d.i.s. has an orthogonal basis.

Proof: Let $\{u_1, u_2, \ldots, u_n\}$ be a basis of an i.p.s. V. We will construct an orthogonal set $\{w_1, w_2, \ldots, w_n\}$ which is a basis of V. The construction is geometric in nature. Therefore just imagine that we are working in $\mathbb{R}^2$.



Let $w_1 = u$. Define $w_2 = u_2 - \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$.

Then $\langle w_2, w_2\rangle = \langle u_1, u_2\rangle - \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle}\langle u_1, u_1\rangle = 0$ & $w_2 \neq 0$

(Note that if $w_2 = 0$, then $\{u_1, u_2\}$ becomes L.D.)

Let us now construct w_3:

Let $w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle u_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$

Then $\langle w_3, w_1\rangle = \langle w_3, w_2\rangle = 0$ & $w_3 \neq 0$.

Proceeding as above by induction, we define

$$w_k = u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

$\Rightarrow \{w_1, \ldots, w_n\}$ is orthogonal, hence it is L.I. Therefore it is a basis. ∎

The process which we used in the proof of the previous result is called Gram-Schmidt Orthogonalization Process

<u>Example:</u> Let us find an orthogonal basis for $\mathbb{P}_2[-1,1]$ starting from $\{1, x, x^2\}$ w.r.to the usual i.p.

Take $w_1 = 1$. Then $w_2 = u_2 - \frac{\langle u_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx}$

$$= x - \frac{\frac{x^2}{2}\Big|_{-1}^{1}}{2} = x - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = x,$$

$$w_2 = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x = x^2 - \frac{\langle x^2, 1 \rangle}{2} - \frac{\langle x^2, x \rangle}{2/3} x$$

$$= x^2 - 1/3 \quad (\text{Check!}).$$

So $\{1, x, x^2 - 1/3\}$ is an orthogonal basis for $\mathbb{P}_2[-1,1]$.

These polynomials are called Legendre Polynomials.

<u>Exercise:</u> Orthogonalize the L.I. set $\{(1,0,1,1), (-1,0,-1,1), (0,-1,1,1)\}$.

<u>Orthonormal:</u> A subset A of an i.p.s. V is said to be orthonormal if it is orthogonal and $\|u\| = 1 \; \forall u \in A$.

<u>Remarks:</u> (i). If A is an orthogonal set of nonzero vectors then $\left\{ \frac{u}{\|u\|} : u \in A \right\}$ is an orthonormal set.

(ii). From G-S. orthogonalization process, we can obtain an orthonormal basis for any f.d.v.s.

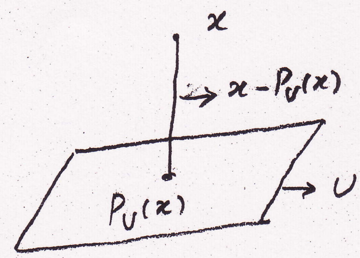
(iii) We will see that an orthonormal basis of a f.d.v.s. acts just like the standard basis of $\mathbb{R}^n$.

Theorem: Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of an ips V. Then for any $x \in V$, $x = \langle x, u_1 \rangle u_1 + \cdots + \langle x, u_n \rangle u_n$.

Proof: Let $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$. Then $\langle x, u_i \rangle = \alpha_i$, $i = 1, 2, \ldots, n$.

Example: $u = (1,2,3) = 1.e_1 + 2e_2 + 3e_3 = \langle u, e_1 \rangle e_1 + \langle u, e_2 \rangle e_2 + \langle u, e_3 \rangle e_3$.

Note that $(1,2,0) = \langle u, e_1 \rangle e_1 + \langle u, e_2 \rangle e_2$ is the projection of $(1,2,3)$ on the xy-plane. We use this idea to define the projection of an element to a subspace of an ips.



orthogonal projection: Let U be a subspace of an ips V and let $\{u_1, u_2, \ldots, u_m\}$ be an orthonormal basis for U. The orthogonal projection $P_U(x)$ of $x \in V$ onto U is defined by:

$$P_U(x) = \langle x, u_1 \rangle u_1 + \cdots + \langle x, u_m \rangle u_m.$$

Note that $P_U(x) \in U$ and $\langle x - P_U(x), u \rangle = 0$ for all $u \in U$, ie, $x - P_U(x)$ is orthogonal to all the elements of U, because $\langle x - P_U(x), u_i \rangle = 0$ $\forall i$.

orthogonal subspaces: Let U and W be subspaces of an ips V. Then U & W are said to be orthogonal, denoted by $U \perp W$, if $\langle u, w \rangle = 0$ for all $u \in U$ & $w \in W$.

orthogonal complement: Let U be a subspace of V. Then $U^{\perp} := \{v \in V : \langle u, v \rangle = 0 \ \forall u \in U\}$ is called orthogonal complement of U.

Remark: (i) $U^{\perp}$ is a subspace of V; infact U & $U^{\perp}$ are orthogonal.

(ii) $x - P_U(x) \in U^{\perp}$.

(iii) $U \cap U^{\perp} = \{0\}$ because if $u \in U \cap U^{\perp}$ then $\langle u, u \rangle = 0$ ie $u = 0$.

(iv) The orthogonal projection of U, $P_U(x)$ is always unique and it is independent of the choice of an orthonormal basis of U. If $P_U(x)$ & $P_U'(x)$ are projections of x over U, then it is clear that $P_U(x) - P_U'(x) \in U$ & $P_U(x) - P_U'(x) = (x - P_U'(x)) - (x - P_U(x)) \in U^{\perp}$

$\Rightarrow P_U(x) - P_U'(x) = 0$ by (iii).