Lecture 16

Example 2: (Matrix with repeated eigenvalues but full eigenvectors):

$$\text{Let } A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Check 1: The ch. polynomial $\det(A - \lambda I) = 0$ is $(1-\lambda)(5-\lambda)^2 = 0$

$\Rightarrow \lambda_1 = 1$ and $\lambda_2 = 5$ are the eigenvalues.

Check 2: Eigen space corresponding to $\lambda_1 = 1$: Solve $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- We get $x_1 = (1,1,0)$ is an eigenvector.
- Here the rank of the matrix is 2; hence the nullity is 1.
- Therefore, the eigenspace $E(1) = \{t(1,1,0) : t \in \mathbb{R}\}$.

Check 3: Eigen space corresponding to $\lambda_2 = 5$: Solve $\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Note that here the rank is 1; hence the nullity is 2.

Since $x_1 + x_2 = 0$ & $x_3 \in \mathbb{R}$, $E(5) = \text{span}\{(-1,1,0), (0,0,1)\}$.

Example 3: (Matrix with repeated eigenvalues with insufficient eigenvectors)

$$\text{Let } A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Check 1: The ch polynomial is $(3-\lambda)^5 = 0$ & $\lambda = 3$ is the only eigen value - repeated 5 times (ie with multiplicity 5)

Check 2: Solve $(A - 3I)x = 0$. We get the rank of $A - 3I$ is 4; hence the nullity is 1. Moreover, $x_2 = x_3 = x_4 = x_5 = 0$.

$\Rightarrow (1,0,0,0,0)$ is an eigenvector and the corresponding eigenspace is $\{t(1,0,0,0,0) : t \in \mathbb{R}\}$.

Example 4: If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then the ch equation is $x^2 + 1 = 0$ which has complex roots. So the eigenvalues of A are complex.

Let us derive some properties of the eigenvalues.

Lemma: If A is a triangular matrix, then the diagonal entries are exactly the eigenvalues.

Proof: Let A be an upper triangular matrix. Then

$$\det(A-\lambda I) = \det\begin{bmatrix} a_{11}-\lambda & * & * & \cdots & * \\ & \ddots & & \vdots & \\ & & & * & * \\ 0 & & & & a_{nn}-\lambda \end{bmatrix} = (a_{11}-\lambda)\cdots(a_{nn}-\lambda)=0$$

Theorem: Let A be an $n\times n$ matrix. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues possibly with repetition. Then

1. $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$
2. $\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Proof: The proofs will be discussed in the tutorial class.

Diagonalization of matrices:

Definition: A square matrix is said to be diagonalizable if $\exists$ an invertible matrix Q s.t. $Q^{-1}AQ$ is a diagonal matrix.

If A is diagonalizable then there exists a diagonal matrix D such that $A = QDQ^{-1}$ for some invertible matrix Q. We will discuss under what conditions on A, the matrix is diagonalizable and how to find Q & D. We will also see some applications.

The next result characterizes a diagonal matrix, and the proof reveals a practical way to find Q & D.

Theorem: Let A be an $n\times n$ matrix. Then A is diaganizable if and only if A has n L.I. eigen-vectors.

Proof: Suppose $Ax_i = \lambda_i x_i$, $i=1,2,\ldots,n$ and $\{x_1,\ldots,x_n\}$ is L.I.

Define $Q = [x_1\ x_2 \ldots x_n]$. Then $AQ = [Ax_1\ Ax_2 \ldots Ax_n]$
$= [\lambda_1 x_1\ \lambda_2 x_2 \cdot \lambda_n x_n] = QD$

where D is the diagonal matrix having $\lambda_1, \lambda_2, \ldots \lambda_n$ on the diagonal.

Since the column vectors are L.I, φ is invertible.

Therefore $\varphi^{-1}A\varphi$ is a diagonal matrix.

Converse: Suppose $\varphi^{-1}A\varphi = D$, ie $A\varphi = \varphi D$. Let $x_1, \dots, x_n$ be the column vectors of φ. Then, as seen above, we have

$$A\varphi = [Ax_1 \; Ax_2 \dots Ax_n] = [\lambda_1 x_1 \dots \lambda_n x_n] = \varphi D,$$

$$\text{ie } Ax_i = \lambda_i x_i.$$

Since φ is invertible, $x_1, x_2, \dots, x_n$ are L.I. $\blacksquare$

Example1. Let $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

We have already seen that $\lambda_1 = 1$ & $\lambda_2 = 5$ are the eigenvalues & $E(1) = \text{span}\{(1,1,0)\}$ & $E(5) = \text{span}\{(-1,1,0),(0,0,1)\}$.

Note that $\{(1,1,0),(-1,1,0),(0,0,1)\}$ is L.I. Take

$\varphi = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, we get $A = \varphi D \varphi^{-1}$ or $\varphi^{-1}A\varphi = D$.

Example 2: The matrix $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable and we can't find 5 L.I

as $E(\lambda = 3) = \text{span}\{(1,0,0,0,0)\}$

e-vectors.