Lecture 17

We now discuss an application of the diagonalization. Consider the Fibonacci numbers or sequence: $F_0 = 0$, $F_1 = 1$ & $F_{n+1} = F_{n+1} + F_n$ or $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ This sequence has many beautiful Mathematical properties and there is even a scholarly journal "Fibonacci Quarterly" devoted to articles about the sequence & related topics. On many plants the leaves grow in a spiral pattern and the number of leaves (in the stages) are Fibonacci numbers or can be defined by these numbers. The seeds in a sunflower has a pattern which can be defined by Fibonacci numbers.

<u>Binet Formula:</u> There is a formula called Binet formula to evaluate the nth Fibonacci number. We use the diagonalization to derive this formula.

<u>Our system:</u>

Note that
$$\left.\begin{array}{l} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{array}\right\} \iff (F_{k+2}, F_{k+1}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$
$$\iff u_{k+1} = A u_k$$

where $u_0 = (F_1, F_0) = (1, 0)$.

Moreover, $u_{k+1} = A u_k = A^{k+1} u_0$.

<u>Problem:</u> We have basically converted the original problem into a problem involving a matrix. We now have to compute A^k!

Let us diagonalize A. Note that $|A - \lambda I| = 0 \iff \lambda^2 - \lambda - 1 = 0$. Therefore the eigen-values are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Since the second row of $A - \lambda I$ is $(1, -\lambda)$, the eigen-vectors are $(\lambda_1, 1)$ and $(\lambda_2, 1)$. Therefore, $A = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} = \varphi D \varphi^{-1}$.

Using Jordan method or other method, we see that $\varphi^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & * \\ -1 & * \end{bmatrix}$.

(we don't need the second column)

Recall that $u_k = A^k u_0 = \varphi D^k \varphi^{-1} u_0$. Therefore $u_k = \varphi D^k \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$

$$\Rightarrow u_k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$\Rightarrow u_k = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k \\ -\lambda_2^k \end{bmatrix} = \frac{1}{\sqrt{5}} \left[\lambda_1^{k+1} - \lambda_2^{k+1}, \; \lambda_1^k - \lambda_2^k \right].$$

Since $u_k = (F_{k+1}, F_k)$, $F_k = \frac{1}{\sqrt{5}} \left[\lambda_1^k - \lambda_2^k \right]$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right]$$

— Binet formula. ▨

We have seen that if A is an $n \times n$ matrix and has n L.I eigen-vectors, then A is diagonalizable. The following result assures when we will have n L I eigenvectors.

<u>Theorem</u>: If $v_1, v_2, \dots, v_k$ are eigen-vectors of A corresponding to k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ then they are L.I. set of vectors.

<u>Proof</u>: Suppose $\{v_1, v_2, \dots, v_k\}$ be L.D. We can assume w.l.o.g. that $\{v_1, v_2, \dots, v_p\}$ is L.I and if we take one more element to this, the new set is L.D. Note that here $p < k$. Since $\{v_1, v_2, \dots, v_{p+1}\}$ is L.D., $\exists\, c_i$ (not all zero) s.t. $c_1 v_1 + \dots + c_p v_p + c_{p+1} v_{p+1} = 0 \;\dots(1)$

$$\Rightarrow c_1 A v_1 + \dots + c_p A v_p + c_{p+1} A v_{p+1} = 0$$

$$\Rightarrow c_1 \lambda_1 v_1 + \dots + c_p \lambda_p v_p + c_{p+1} \lambda_{p+1} v_{p+1} = 0 \;\dots(2)$$

$\Rightarrow c_1(\lambda_1 - \lambda_{p+1}) v_1 + \dots + c_p(\lambda_p - \lambda_{p+1}) v_p = 0$ (by taking $(2) - \lambda_{p+1}(1)$)

Since eigenvalues are distinct & $\{v_1, \dots, v_p\}$ is L I, $c_1 = c_2 = \dots = c_p = 0$.
This implies $c_{p+1} = 0 \Rightarrow \{v_1, v_2, \dots, v_{p+1}\}$ is L I. $\Rightarrow\Leftarrow$ ▨

<u>Cor:</u> If A is an $n \times n$ matrix and it has n distinct eigenvalues then A is diagonalizable.

<u>Examples</u>: The matrices $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ & $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix}$ are diagonalizable as each one has distinct eigenvalues: 0, 3 for A & 1, 5 & 7 for B.

In the next result we will prove that a (real) symmetric matrix is diagonalizable. Observe that B is not symmetric.