Lecture 18

In the previous lecture, we have seen that if an $n\times n$ matrix has n distinct eigenvalues then the matrix is diagonalizable. In this lecture we will see that any (real) symmetric matrix is diagonalizable.

We will first see that a square matrix whose row vectors or column vectors are orthonormal has some interesting properties.

The proof of the following lemma will be discussed in the Tutorial class.

Lemma: Let A be $n\times n$ matrix. Then the following statements are equivalent.

1. The column vectors are orthonormal
2. $A^TA = I_n$
3. $A^T = A^{-1}$
4. $AA^T = I_n$
5. $\|Ax\| = \|x\| \quad \forall x \in \mathbb{R}^n$
6. $\langle Ax, Ay\rangle = \langle x, y\rangle, \quad \forall x, y \in \mathbb{R}^n$
7. The row vectors are orthonormal

Definition: A square matrix A is called an orthogonal matrix if A satisfies one of the statements of the above lemma, in particular, $AA^T = A^TA = I$.

Examples: The matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$ & $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are orthogonal.

So far we have been dealing with matrices whose entries are real.

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If A is a complex matrix, then we can define eigen-values & eigen-vectors and do the diagonalization as we did for the real case. In the complex case, the eigenvalues are in $\mathbb{C}$ & the eigen-vectors are in $\mathbb{C}^n$. In $\mathbb{C}^n$, we define $\langle\, ,\rangle$ as follows: $\langle u, v\rangle = u\bar{v}^t$.

Theorem: Let A be an $n\times n$ (real) matrix. If A is symmetric, then A has n real eigenvalues.

Proof: The characteristic polynomial $|A-\lambda I|$ has n roots in $\mathbb{C}$ & each root is an eigenvalue of A.

Let $\lambda \in \mathbb{C}$ be any eigenvalue & $u \in \mathbb{C}^n$ be a corresponding eigenvector of A. Then $Au = \lambda u$ ie $u^tA = \lambda u^t$ (as $A^t = A$)

$\Rightarrow \bar{u}^tA = \bar{\lambda}\bar{u}^t$ (by taking complex conjugate on both sides)

$\Rightarrow \bar{u}^tAu = \bar{\lambda}\bar{u}^tu$, but we know $\bar{u}^tAu = \lambda\bar{u}^tu$

$\Rightarrow \bar{\lambda}\bar{u}^tu = \lambda\bar{u}^tu$ ie $\bar{\lambda}\|u\|^2 = \lambda\|u\|^2$ ie $\lambda = \bar{\lambda}$

Remark: If A is a real symmetric matrix then the eigenvectors are in $\mathbb{R}^n$. This can be proved as follows. Let λ be an e.v. Since it is real & $(A-\lambda I)$ is real and not invertible, $\exists\, u \in \mathbb{R}^n$ s.t. $(A-\lambda I)u = 0$.

Theorem: Let A be a real symmetric matrix. Then $\exists$ an orthogonal matrix Q s.t. $A = QDQ^{-1}$ where D is a diagonal matrix and the diagonal entries of D are the eigenvalues of A.

Proof (*): Let A be $n\times n$ real symmetric matrix. We prove the theorem by induction on n. For $n=1$ the result is obvious. Let us assume that the result is true for all $(n-1)\times(n-1)$ matrices. Let A be an $n\times n$ matrix. By the previous theorem, A has a real eigenvalue; call it λ_1. Let $AX_1 = \lambda X_1$ & $\|X_1\| = 1$. Let $\{X, V_2, \dots, V_n\}$ be an O.N. basis of $\mathbb{R}^n$. This can be obtained by G-S process.

Define $\varphi_1 = [X_1\ V_2 \ldots V_n]$. Note that φ_1 is orthogonal.

Moreover, $\varphi_1^{-1} A \varphi_1$ is a real symmetric matrix because,

$$(\varphi_1^{-1} A \varphi_1)^t = (\varphi_1^t A \varphi_1)^t = (\varphi_1^t A \varphi_1) = \varphi_1^{-1} A \varphi_1.$$

Let us evaluate first column of $\varphi_1^{-1} A \varphi_1$. By symmetry we will know the first row.

The first column is given by

$$(\varphi_1^{-1} A \varphi_1)(e_1) = (\varphi^{-1} A)\varphi_1 e_1 = (\varphi_1^{-1} A) X_1 = \varphi_1^{-1}(\lambda_1 X_1) = \lambda_1 \varphi_1^{-1} X_1 = \lambda_1 e_1$$

because $\varphi_1 e_1 = X_1$

Therefore,

$$\varphi_1^{-1} A \varphi_1 = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_1 \end{array}\right]$$ where A_1 is an $n-1 \times n-1$ square symmetric matrix.

By induction, $\exists$ an orthogonal matrix φ_2 s.t. $\varphi_2^{-1} A \varphi_2 = D_1$ - an $n-1 \times n-1$ diagonal matrix.

<u>Claim:</u> $\exists$ φ such that $\varphi^{-1} A \varphi = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D \end{array}\right]$.

Let us find φ. Note that

$$\left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_1 \end{array}\right] = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & \varphi_2^{-1} A \varphi_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varphi_2^t \end{array}\right] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right]$$

$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varphi_2^t \end{array}\right] (\varphi_1^{-1} A \varphi_1) \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right]$$

$$= \left(\varphi_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right]\right)^t A \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right].$$

One can easily verify that the matrix $\varphi = \varphi_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varphi_2 \end{array}\right]$ is an orthogonal matrix. ▨