Lecture 2

Elementary matrices and elementary row operations:

Let us denote the identity $m \times m$ matrix by I_m. We define the following matrices.

1. $E_i(c)$, $c \in \mathbb{R}\setminus\{0\}$ obtained by multiplying the ith row of I_m by c.
2. E_{ij} - obtained by interchanging ith & jth row of I_m.
3. $E_{ij}(c)$, $c \in \mathbb{R}\setminus\{0\}$ obtained by adding c times the jth row of I_m to its ith row.

<u>Ex:</u> For $m=2$, $E_2(-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ & $E_{21}(\alpha) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ - Change on the second row!

If A is an $m \times n$ matrix & E is an elementary matrix (from I_m), then EA is obtained from A by doing the corresponding row operation on A.

<u>Ex:</u> If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, to get $\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$, we consider

$$EA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}.$$

Similarly, we have elementary column operations (of three types). Performing a particular column operation on a given $m \times n$ matrix is same as post multiplying the given matrix by the $n \times n$ matrix obtained by performing the same operations on the $n \times n$ identity matrix.

<u>Ex:</u> Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ & $B = \begin{pmatrix} a_{11} - a_{12} & a_{12} & a_{13} \\ a_{21} - a_{22} & a_{22} & a_{23} \end{pmatrix}$. Then

$$A \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = B.$$

So analogous to $E_i(c)$, E_{ij} & $E_{ij}(c)$ we have $F_i(c)$, F_{ij} & $F_{ij}(c)$ (obtained by performing column operations to I_n).

<u>Remark:</u> Note that $E_{ij} = F_{ij}$, $E_i(c) = F_i(c)$ but $E_{ij}(c) = F_{ji}(c)$.

<u>Invertible matrix:</u> we say that an $n \times n$ square matrix A is invertible if $\exists$ a matrix B of the same order s.t $AB = I_n = BA$ and in this case B is denoted as A^{-1}.

<u>Exercise:</u> 1. Inverse of a matrix is unique.

2. $(AB)^{-1} = B^{-1}A^{-1}$.

<u>Lemma:</u> An elementary matrix is invertible with elementary inverse.

<u>Proof:</u> Verify that $E_i(c)\cdot E_i(\frac{1}{c}) = I_m = E_i(\frac{1}{c})E_i(c)$,

$E_{ij}E_{ij} = I_m$ & $E_{ij}(c)E_{ij}(-c) = I_m = E_{ij}(-c)\cdot E_{ij}(c)$.

The main application of elementary matrices appears in the following ▨.

<u>Theorem:</u> Let A be an $m\times n$ matrix. Then by applying a sequence of row and column operations A can be reduced to the following form:

$$\begin{pmatrix} I_r & O_{r\times n-r} \\ O_{m-r\times r} & O_{m-r\times n-r} \end{pmatrix}_{m\times n}$$ – where O is a zero matrix.

<u>Proof:</u> If $a_{ij}\neq 0$, interchange 1st & jth column and then 1st & ith rows. Multiply the first row by $\frac{1}{a_{ij}}$ and do the column and row operation to obtain the form $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ 0 & b_{m2} & \cdots & b_{mn} \end{pmatrix}$. Repeat the process. ▨

<u>Theorem:</u> Let A be an $n\times n$ matrix. Then A is invertible iff A is a product of elementary matrices.

<u>Proof:</u> Let A be invertible. By previous lemma $\exists$ E_i's & F_j's s.t.

$E_1E_2\cdots E_p A F_1F_2\cdots F_q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Since the LHS is invertible, the RHS is invertible and hence $r=n$. Therefore, $A = E_p^{-1}E_{p-1}^{-1}\cdots E_1^{-1} I_n F_q^{-1}\cdots F_1^{-1}$.

The converse is obvious. ▨

<u>Theorem:</u> An $n\times n$ matrix A is invertible iff by a sequence of elementary row operations A can be reduced to I_n.

<u>Proof:</u> Suppose A is invertible. Then by previous lemma $A = E_1\cdots E_r$, for some E_i's. Hence $E_r^{-1}\cdots E_1^{-1}A = I_n$, i.e., by row operations, A can be reduced to I_n. Conversely, if $E_1\cdots E_k A = I_n$, then $A = E_k^{-1}\cdots E_1^{-1}$ and therefore A is invertible ▨

<u>Gauss - Jordon Method or Method of row reduction to find A^{-1}</u>

Suppose A is invertible. Then by the previous lemma, $\exists$ $E_1,\ldots,E_r$ s.t. $E_1E_2\cdots E_rA = I$. This implies that $A = E_r^{-1}\cdots E_1^{-1}I_n$

$\Rightarrow A^{-1} = E_1\cdots E_r I_n$.

This means that the sequence of row operations which reduces A to I_n also reduces I_n to A^{-1} (if performed in the same order).

<u>Example:</u> Let $A = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$. To obtain A^{-1}, we use the method given above.

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{ie } E_1(\frac{1}{2})]{\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[E_{31}(5)]{E_{21}(2)} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & 3 & 5 & 1 & 1 & 2 \\ 0 & 5 & \frac{17}{2} & \frac{5}{2} & 0 & 6 \end{array}\right)$$

$$\Big\downarrow \text{after a few steps}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & -3 \\ 0 & 1 & 0 & -8 & 17 & -10 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right)$$

$\uparrow A^{-1}$.

<u>Example:</u> Let $A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix}$. In this case,

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 6 & 0 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{few operations}]{\text{after a}} \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & -1 & 1 \end{array}\right)$$

Note that if A is invertible then EA is invertible, for any elementary matrix A. Hence A is not invertible.