Lecture 3

Determinants: There are different ways to define determinants. Each one has its advantages. We will define using the concept permutation.

Permutation: Let S be any finite set. Here we take $S=\{1,2,\dots,n\}$. Any one-one onto mapping of S to itself is called permutation. If σ is a permutation then σ^{-1} is also a permutation. Also, given a permutation σ and τ, the composition $\sigma\circ\tau$ is also a permutation.

We denote the set of all permutations on $\{1,2,\dots,n\}$ by S_n.

Examples: Let σ & $\varphi \in S_4$ be defined by $\sigma=\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}$ & $\varphi=\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}$. Then $\sigma\circ\varphi=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$ & $\varphi\circ\sigma=\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}$ & $\sigma^{-1}=\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}$. Clearly $\sigma\sigma^{-1}=\sigma^{-1}\sigma=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$.

Transposition: A permutation is called transposition if it moves exactly two points in S.

We need the following two results which we state without proof.

Theorem: 1. Every permutation can be written as a product (composition) of transpositions.

2. If σ is a permutation and $\sigma_1,\sigma_2,\dots,\sigma_r$ and $\tau_1,\tau_2,\dots,\tau_s$ are transpositions s.t. $\sigma=\sigma_1\circ\sigma_2\circ\dots\circ\sigma_r$ and $\sigma=\tau_1\circ\tau_2\circ\dots\circ\tau_s$ then r & s are both even or odd. (Note that here σ_i's or τ_i's need not be distinct)

Ex: Let $\sigma=\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}$. Then $\sigma=\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}\circ\begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}\circ\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}$.

We write $\sigma=(1\,3)(1\,4)(1\,2)$. Note that $\sigma=(2\,1)(2\,3)(2\,4)$

Even or odd permutation: A permutation σ is called an even permutation if it can be written as a product of an even number of transpositions. Otherwise it is called an odd permutation.

The identity permutation I is even and every transposition is odd.

Define $\text{sign}(\sigma) = +1$ if σ is even & $\text{sign}(\sigma) = -1$ if σ is odd.

Determinants: Let $A = (a_{ij})$ be an $n\times n$ matrix. Its determinant, denoted by $|A|$ is:

$$\det A = |A| = \sum_{\sigma\in S_n} \text{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

(Note that S_n has $n!$ elements).

Example: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$. Here $S_2 = \{I, (1,2)\}$. So

$$|A| = (\text{sign } I)\, a_{11} a_{22} + \text{sign}(1,2)\, a_{12} a_{21} = a_{11}a_{22} - a_{12}a_{21}.$$

This definition is not very convenient for computing determinants. However, several properties of determinants can be derived easily from this definition.

Properties of the determinants: (P 1): Let $A = (a_{ij})$ & $B = (b_{ij})$ be $n\times n$ matrices. Then if B is obtained by interchanging two rows of A then $|A| = -|B|$.

Proof: suppose $b_{pj} = a_{qj}$, $b_{qj} = a_{pj}$ & $a_{ij} = b_{ij}$ $\forall i \neq p, q$ & $\forall j$.

Consider the transposition $\tau = (p\ q)$. Note that $S_n = \{\sigma\circ\tau : \sigma\in S_n\}$.

Hence
$$|B| = \sum_{\sigma\in S_n} \text{sign}(\sigma\circ\tau)\, b_{1\sigma\tau(1)} b_{2\sigma\tau(2)} \cdots b_{p\sigma\tau(p)} \cdots b_{q\sigma\tau(q)} \cdots b_{n\sigma\tau(n)}$$
$$= \sum_{\sigma\in S_n} \text{sign}(\sigma)\,\text{sign}(\tau)\, b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{p\sigma(q)} \cdots b_{q\sigma(p)} \cdots b_{n\sigma(n)}$$
$$= (\text{sign } \tau) \sum_{\sigma\in S_n} \text{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{q\sigma(q)} \cdots a_{p\sigma(p)} \cdots b_{n\sigma(n)}$$
$$= -|A|. \ \blacksquare$$

Cor: (P2) If A has two identical rows then $|A| = 0$.

Proof: Let B be the matrix obtained by interchanging those two identical rows. Then $A = B$. Since $|A| = -|B|$ by P_1, $|A| = 0$. ∎

The proofs of the following properties P_3 & P_4 are similar to the proof of (P_1).

Prop: (P3) If B is obtained by multiplying a row of A by a constant c, then $|B| = c|A|$.

Prop: (P4) Suppose $C = (c_{ij})$. Further assume that A, B and C differ only in the k th row for some k s.t. $c_{kj} = a_{kj} + b_{kj}\ \forall j$. Then $|C| = |A| + |B|$.

Prop: (P5) If B is obtained by adding c times the pth row of A to its q th row then $|A| = |B|$.

Proof: Note that $b_{qj} = a_{qj} + c\, a_{pj}\ \forall j$ and $b_{ij} = a_{ij}\ \forall i \neq q$.
Let $C = (c_{ij})$ where, for all j, $c_{ij} = a_{ij}$ if $i \neq q$ and $c_{qj} = c\, a_{pj}$.
Then A, B and C differ only the qth row and $b_{qj} = a_{pj} + c_{qj}$.
Hence by (P4), $|B| = |A| + |C|$. Further $|C| = 0$ by (P_3) & (P_2).