Lecture 4

Proposition: (P6) If E is an elementary matrix then $|EA| = |E||A|$.

Proof: We have already seen that $|E_{ij}| = -1$, $|E_{ij}(c)| = 1$ and $|E_i(c)| = c$. Hence $|EA| = |E||A|$.

Proposition: (P7) A is not invertible iff $|A| = 0$.

Proof: Suppose A is invertible. Then $A = E_1 E_2 \cdots E_r$ for some elementary matrices E_i's. Therefore $|A| \neq 0$.

Suppose A is not invertible. Then $E_1 E_2 \cdots E_p A F_1 \cdots F_q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $r < n$ for some E_i's & F_j's. This implies that

$$A = E_p^{-1} \cdots E_1^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} F_q^{-1} \cdots F_1^{-1} = E_p^{-1} \cdots E_1^{-1} D \quad \cdots\cdots (*)$$

where D is a matrix whose last row is zero. Therefore, by the definition, $|A| = 0$.

Proposition: (P8): $|AB| = |A||B|$.

Before proving this let us prove the following result:

Proposition: Let A be an $n \times n$ matrix s.t. $AB = I$. Then A is invertible.

Proof: If A is not invertible, then by $(*)$ $\exists$ E_i's s.t. $A = E_1 \cdots E_p D$ where D is a matrix whose last row is zero. Therefore, $AB = (E_1 \cdots E_p D) B = I$, i.e., $DB = (E_1 \cdots E_p)^{-1} I$ where the last row of DB is zero. Since $\det(\text{LHS}) = 0$, $\det(\text{R.H.S.}) = 0$ which is a contradiction.

Cor: Let A be an $n \times n$ matrix. If A is not invertible, then AB is not invertible for any $n \times n$ matrix B.

Proof: Exercise.

Let us come back to the proof of $|AB| = |A||B|$.

Proof: If A is not invertible, then by previous result $0 = |AB| = |A||B| = 0$.

Suppose A is invertible. Then $A = E_1 \cdots E_r$ for some E_i's.
Therefore, by (P6), $|AB| = |E_1 \cdots E_r B| = |E_1| \cdots |E_r| |B|$.

__Proposition:__ __(P9)__ : $|A| = |A^t|$.

__Proof:__ Exercise

__Remark:__ The properties (P1) – (P9) are valid if the word row is replaced by the word column.