Lecture 5

As mentioned earlier, evaluating a determinant from the definition is not easy. We will derive two more properties of the determinant which will provide an inductive method for computing the determinants.

Lemma: Let $A = \begin{pmatrix} B & \begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \end{matrix} \\ 0\,0 \cdots 0 & 1 \end{pmatrix}$ where $B = (a_{ij})_{n-1 \times n-1}$. Then $|A| = |B|$.

Proof: By definition $|A| = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$

$$= \sum_{\sigma \in S_n, \sigma(n)=n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \quad \left(\because \sigma(n) \neq n \text{ then } a_{n\sigma(n)} = 0\right)$$

$$= \sum_{\sigma \in S_n, \sigma(n)=n} (\operatorname{sign}\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n-1\,\sigma(n-1)}$$

$$= |B| \quad \left(\because S_{n-1} = \{\sigma \in S_n : \sigma(n) = n\}\right).$$

Theorem: Let $A = (a_{ij})$ and let A_{ij} denote the matrix obtained by removing the ith row and the jth column of A. Then
$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|.$$

Proof: Fix an $i \in \{1, 2, \ldots, n\}$. Define

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & 0 & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ 0 & a_{i2} & 0 \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad A_3 = \ldots, A_n = \ldots$$

Note that $a_{ij} = (A)_{ij} = (A_1)_{ij} + (A_2)_{ij} + \cdots + (A_n)_{ij}$ and other rows of matrices $A, A_1, A_2, \ldots, A_n$ are same. Hence by (P4), $|A| = \sum_{j=1}^{n} |A_j|$.

Claim: $|A_j| = (-1)^{i+j} a_{ij} |A_{ij}|$, where A_{ij} is as defined in the statement of the result.

Note that

$$A_j = \begin{pmatrix} a_{11} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \\ 0 & \cdots & a_{ij} & \cdots & 0 \\ \vdots & & & & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \xrightarrow[\substack{(n-j)\text{ column} \\ \text{operations}}]{\substack{\text{after} \\ (n-i)\text{ row \&}}} \begin{pmatrix} & A_{ij} & & * \\ & & & * \\ & & & * \\ & & & \vdots \\ 0 & 0 & \cdots 0 & a_{ij} \end{pmatrix}$$

$$\Rightarrow |A_j| = (-1)^{2n-i-j} \begin{vmatrix} A_{ij} & * \\ 0 & a_{ij} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} A_{ij} & * \\ 0 & 1 \end{vmatrix}$$

$$= (-1)^{i+j} a_{ij} |A_{ij}| \text{ (by previous lemma).}$$

<u>Determinant method of finding inverse:</u>

Take some $k \neq i$ and obtain the matrix B by replacing the ith row of A with the kth row of A (by keeping the kth row as it is). Then we get, $\sum_{j=1}^{n} (-1)^{i+j} a_{kj} |A_{ij}| = |B| = 0$ (as two rows of B are same).

So, if we set $C_{ij} = (-1)^{i+j} |A_{ij}|$, then we have the following n equations:

$$\sum_{j=1}^{n} a_{ij} C_{ij} = |A| \quad \text{and} \quad \sum_{j=1}^{n} a_{kj} C_{ij} = 0.$$

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{i1} \\ C_{i2} \\ \vdots \\ C_{in} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ |A| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & \cdots & C_{i1} & \cdots & C_{n1} \\ C_{12} & \cdots & C_{i2} & \cdots & C_{n2} \\ \vdots & & \vdots & & \vdots \\ C_{1n} & \cdots & C_{in} & \cdots & C_{nn} \end{pmatrix} = |A| I_n$$

ie $A (C_{ij})^t = |A|$, The matrix (C_{ij}) is called the matrix of cofactor of A

The matrix $(C_{ij})^t$ is called the (classical) adjoint (or adjunct) of A and it is denoted by Adj A. Thus

$$A(\mathrm{Adj}\,A) = |A|\, I_n.$$

Therefore the following result is immediate from this fact.

<u>Theorem:</u> Let A be an $n\times n$ matrix. If A is invertible, then

$$A^{-1} = \frac{1}{|A|}(\mathrm{Adj}\,A).$$

<u>Cramer's Rule for solving system of linear equations:</u>

The following result is about solvability of a system of linear equations.

<u>Theorem:</u> Let A be an $n\times n$ matrix. Then the following statements are equivalent:

1. $|A| \neq 0$
2. A is invertible
3. $Ax = b$ has a unique solution for every ($n\times 1$ matrix) b
4. $Ax = b$ has a solution for every b

<u>Proof:</u> (1) $\Leftrightarrow$ (2): we have already seen the proofs of these implications.

(2) $\Rightarrow$ (3): For b, choose $x = A^{-1}b$ and note that $Ax = A(A^{-1}b) = b$ and $A^{-1}b$ has to be the only solution for $Ax = b$.

(3) $\Rightarrow$ (4): obvious.

(4) $\Rightarrow$ (2): For $b_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ - 1 at the ith place, $\exists\, v_i$ s.t. $Av_i = b_i$:

Hence $AB = I_n$, where $B = (v_1\, v_2 \ldots v_n)$.

<u>Cor:</u> Let A be an $n\times n$ matrix. Then the following are equivalent:

1. A is invertible
2. $Ax = 0$ has only the trivial solution $x = 0$.

Proof: (1) $\Rightarrow$ (2) follows from the previous result.

(2) $\Rightarrow$ [(3) of the previous result]: Suppose $Au_1 = Au_2 = b$ for some b and $u_1 \neq u_2$. Then $A(u_1 - u_2) = 0$ where $u_1 - u_2 \neq 0$ which is a contradiction.

Cramer's Rule: Let $Ax = b$ be a system of n linear equations in n unknowns s.t. $|A| \neq 0$. Then the system has a unique solution given by

$$x_j = \frac{|C_j|}{|A|}, \quad j = 1, 2, \ldots, n.$$

where C_j is the matrix obtained from A by replacing the jth column with the column matrix $b = (b_1\, b_2 \ldots b_n)^t$.

Proof: If $|A| \neq 0$, then A is invertible and $x = A^{-1}b$ is the unique solution of $Ax = b$,

i.e., $$x = \frac{1}{|A|}(\operatorname{adj} A)\, b$$

$$\Rightarrow \quad x_j = \frac{1}{|A|} b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$$

$$= \frac{|C_j|}{|A|} \quad \blacksquare$$