Lecture 6

We have seen that if $|A| = 0$ and the system $Ax = 0$ has more than one solution then it has infinite number of solutions. We will see that, in this case, the set of solutions has a structure, called a vector space. We will first introduce the notion of a vector space which is is generalization of the algebraic structure present in $\mathbb{R}^3$. The study of matrices will be elaborated within this framework.

Definition: A real vector space is a nonempty set V with two algebraic operations that satisfy the following rules:

(A) There is an operation called addition that associates to every pair of elements $x, y \in V$ a unique element $x + y \in V$ s.t.

(i) $x + y = y + x$ (ii) $x + (y + z) = (x + y) + z$

(iii) $\exists$ a unique element in V, called 0, s.t $x + 0 = 0 + x = x$

(iv) for any $x \in V$, $\exists$ an element $-x \in V$ s.t. $x + (-x) = (-x) + x = 0$

(0 is called additive identity and $-x$ the additive inverse of x).

(B) There is an operation called scalar multiplication that associates to each $x \in V$ and $\alpha \in \mathbb{R}$ a unique element $\alpha x \in V$ s.t.

(v) $\alpha(x + y) = \alpha x + \alpha y$ (vi) $\alpha(\beta x) = (\alpha\beta)x$

(vii) $(\alpha + \beta)x = \alpha x + \beta y$ (viii) $1 \cdot x = x \;\; \forall x \in V$

The elements of a vector space are called vectors.

Examples:

1. The set of reals $\mathbb{R}$ is a vector space.

2. Let $V = \mathbb{R}^n = \{(x_1, x_2, \ldots x_n) : x_i \in \mathbb{R}, 1 \le i \le n\}$. Define $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and

for $\alpha \in \mathbb{R}$, $\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$. Then V is a vector space.

3. Let $V = M(n, \mathbb{R})$ - the set of all $n \times n$ matrices with real entries. Define $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ and $\alpha(a_{ij}) = (\alpha a_{ij})$. Then V is a vector space.

4. Let A be an $m \times n$ matrix. Then $V = \{x : Ax = 0\}$ - the set of solutions of $Ax = 0$, is a vector space.

5. Let V be the set of all polynomials with real coefficients and degree less or equal to $n-1$. For $P(x) = \sum_{i=0}^{n-1} a_i x^i$ and $Q(x) = \sum_{i=0}^{n-1} b_i x^i$, $\alpha \in \mathbb{R}$, define $(P+Q)(x) = \sum_{i=0}^{n-1} (a_i + b_i) x^i$ and $(\alpha P)(x) = \sum_{i=0}^{n-1} \alpha a_i x^i$. Then V is a vector space with additive identity as the zero polynomial. Note that as sum of two nth degree polynomial may not be a nth degree polynomial, the set of all nth degree polynomials is not a vector space.

<u>Definition:</u> A complex vector space is where we can choose $\alpha \in \mathbb{C}$ and the scalar multiplication satisfies (v) - (viii).

For example $\mathbb{C}^n$, $M(n, \mathbb{C})$ etc. are complex vector spaces.

From now onwards by V we will mean a real vector space.

<u>Theorem:</u> Let V be a vector space and $x \in V$. Then

1. $0 \cdot x = 0$ 2) Additive identity and additive inverse are unique 3) $(-1)x = -x$ 4) $\alpha \cdot 0 = 0$ $\forall \alpha \in \mathbb{R}$ 5). If $\alpha x = 0$ then either $\alpha = 0$ or $x = 0$.

<u>Proof:</u> We will prove only 1). Others are similar. We have

$$0x = -(0x) + 0x = -(0x) + (0+0)x$$
$$= -(0x) + 0x + 0x = 0 \cdot x \quad \blacksquare$$

Definition: Let W be a nonempty subset of a vector space V. Then W is called a vector subspace (or simply subspace) of V if W is a vector space under the operations defined in V.

In order to show a subset a subspace, there is no need to verify the rules: (i) – (viii) of the vector space. It is enough to check that (i) $0 \in W$ (ii) $w_1 + w_2 \in W$ $\forall w_1, w_2 \in W$ (iii) $\alpha w \in W$ for $\alpha \in \mathbb{R}$ & $w \in W$. The other rules are satisfied automatically.

Examples: 1. The set $A = \{(x, y, z) : x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$. But $B = \{(x, y, z) : x + y + z = 1\}$ is not a subspace of $\mathbb{R}^3$ as $0 \notin B$.

2. The set of all polynomials of degree $\leq n-2$ is a subspace of all polynomials of degree $\leq n-1$.

3. $\{0\}$ is always a subspace of any vector space.

4. Set of all polynomials with nonnegative coefficients & degree $\leq n-1$ is not a subspace of all polynomials of degree $\leq n-1$.

5. The set of points on a straight line which is not passing through origin is not a subspace of $\mathbb{R}^2$.

6. The set of points on a circle is not a subspace of $\mathbb{R}^2$.

7. The union of two straight lines passing through origin is not a subspace (unless both are same).