Linear span:

1. Take $(1,1) \in \mathbb{R}^2$ and consider $\{\alpha(1,1) : \alpha \in \mathbb{R}\}$. This is a straight line passing through origin and hence it is a subspace of $\mathbb{R}^2$.
2. Note that $\{\alpha(1,1) + \beta(1,0) : \alpha, \beta \in \mathbb{R}\} = \{\alpha(1,0) + \beta(0,1) : \alpha \in \mathbb{R}\} = \mathbb{R}^2$.
3. In $\mathbb{R}^3$, $\{\alpha(1,1,1) + \beta(2,1,3) : \alpha, \beta \in \mathbb{R}\}$ is a plane passing through origin and this is a subspace of $\mathbb{R}^3$. (The equation of the plane is $2x - y = z$).

In these examples we see that one or two elements generate subspaces. We will define this concept formally below.

Definition: Let $S = \{u_1, u_2, \ldots u_n\}$ be a subset of a vector space V. The linear span of S is the set defined by

$$\text{span}(S) \text{ or } L(S) = \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n : \alpha_i \in \mathbb{R}, 1 \le i \le n\}.$$

If S is empty, we define $L(S) = \{0\}$. The combination $\alpha_1 u_1 + \cdots + \alpha_n u_n$ is called a linear combination of u_i's.

Example: Is $(4,5,5)$ a linear combination of $(1,2,3)$, $(-1,1,4)$ & $(3,3,2)$.

To answer, one has to find α, β, γ s.t $\alpha(1,2,3) + \beta(-1,1,4) + \gamma(3,3,2) = (4,5,5)$. In fact, $3(1,2,3) + (-1)(-1,1,4) + 0(3,3,2) = (4,5,5)$

If S is any arbitrary subset of V, the linear span is defined as follows:

$$L(S) = \{\alpha_1 u_1 + \cdots + \alpha_k u_k : \alpha_i \in \mathbb{R}, u_i \in S\},$$

the collection of all (finite) linear combination of elements of S.

Proposition: Let S be any nonempty subset of a vector space V. Then $L(S)$ is a subspace of V. In fact, it is the smallest subspace of V containing S. (We say that $L(S)$ is spanned by S).

Proof: It is easy to verify that $L(S)$ is a subspace and $S \subseteq L(S)$. If W is a subspace of V s.t. $S \subset W$, then every linear combination of elements of S belongs to W, i.e. $L(S) \subseteq W$. ∎

Linearly dependent:

Note that span $\{(1,1),(2,2)\}$ is a proper subspace of $\mathbb{R}^2$ but span $\{(1,1),(0,1)\} = \mathbb{R}^2$. In the first case $(2,2) = 2(1,1)$, i.e., $(2,2)$ depends on $(1,1)$ but in the second case one element doesn't depend on the other.

Definition: Let $v \in V$ and $\{v_1, v_2, \dots, v_k\} \subseteq V$. We say that v is linearly dependent (L.D) on $v_1, v_2, \dots, v_k$ if there exist $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ s.t. $v = \sum_{i=1}^{k} \alpha_i v_i$.

Ex: We have seen that $(4,5,5) = 3(1,2,3) + (-1)(-1,1,4) + 0(3,3,2)$.
Note that in this case, $(-1,1,4) = 3(1,2,3) - (4,5,5) + 0(3,3,2)$.
So if v is l.d on $v_1, v_2, \dots v_k$, then $\exists$ i such that v_i is L.D. on $v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots v_k, v$. So in a slightly different point of view we say that $\{v_1, v_2, \dots, v_k, v\}$ is a L.D. set if there exist an element which can be written as a linear combination of the rest of the elements.

Definition: We say that a set $\{v_1, v_2, \dots, v_n\}$ is L.D. if $\exists$ $\alpha_i \in \mathbb{R}$, $1 \le i \le n$, not all zero such that $\sum_{i=1}^{n} \alpha_i v_i = 0$. If $\{v_1, v_2, \dots, v_n\}$ is not L.D. then it is called linearly independent (L.I)

In order to verify a given set $\{v_1, v_2, \dots v_n\}$ is LD or L.I, we consider the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

In case, $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$ is the only solution then the set is L I otherwise it is L.D.

Example: 1. Since $3(1,2,3) - (-1,1,4) - (4,5,5) = 0$, the set $\{(1,2,3), (-1,1,4), (4,5,5)\}$ is L.D

2. Consider the set $\{(2,0,0), (3,1,0), (5,6,4)\}$ and

$\alpha(2,0,0) + \beta(3,1,0) + \gamma(5,6,4) = 0.$

It is easy to verify that $\alpha = \beta = \gamma = 0$. So the given set is L.I.

3. verify that $\{(1,1,1), (1,1,0), (0,0,1)\}$ is L.I.

<u>Proposition:</u> Let V be a vector space and $S \subseteq V$.

1. If S is L.I then $0 \notin S$.
2. If S is L.I, then every non empty subset of S is L.I.
3. If S is L.D, then even set containing S is also L.D.

consider the sets $\{(1,0,0), (0,1,0)\}$ and $\{(1,0,0), (0,1,0), (0,0,1)\}$.

These two sets are L.I. However, the set $\{(1,0,0), (0,1,0), (0,0,1)\}$

is something special because, it spans the entire space $\mathbb{R}^3$.

<u>Definition:</u> A subset $B = \{u_1, u_2, \ldots, u_n\}$ is a basis of V if

(i) B is L.I

(ii) $\text{span}(B) = V$.

Recall that the condition (ii) says that every element of V

can be expressed as a linear combination of elements of B.

<u>Examples:</u> 1. The set $\{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$ is

a basis of V. The basis $\{e_1, e_2, e_3\}$ is called the standard basis for $\mathbb{R}^3$.

2. The set $\{(1,1,0), (0,-1,1), (1,0,1)\}$ is L.D. Hence it is not

a basis of $\mathbb{R}^3$. Note the set can't span $\mathbb{R}^3$.

3. The set $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$ spans $\mathbb{R}^3$ but

it is not a basis, because it is not L.I.

4. The set $\{(1,1,1), (1,1,0), (1,0,0)\}$ is a basis for $\mathbb{R}^3$. This is

different from the standard basis of $\mathbb{R}^3$.

5. The set $\{1, x, x^2\}$ is a basis for $\mathbb{P}_3$, the space of all polynomials

of degree ≤ 2.

<u>Remark:</u> If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, then any $v \in V$ is a unique

linear combination $v_1, v_2, \ldots, v_n$.