Lecture 8

Dimension:

We often say that $\mathbb{R}^1$ is one dimensional, $\mathbb{R}^2$ is two dimensional and $\mathbb{R}^3$ is three dimensional. Note that $\{1\}$ is a basis for $\mathbb{R}^1$, $\{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$ and $\{e_i : i=1,2,3\}$ is a basis for $\mathbb{R}^3$. Similarly $\{e_i = (0,0,\ldots 0,1,0,\ldots 0) : i=1,2\ldots n\}$ is a basis for $\mathbb{R}^n$ which is called n dimensional space. We see that the concept dimension is related to the number of elements in a basis. We define this concept for any vector space below.

We need the following result:

Theorem: If a homogeneous system $Ax = 0$ has more unknowns then equations, then it has a nontrivial solution.

We will see a proof later. However, this result can also be proved using the idea of Gauss elimination method. We illustrate with an example.

Example: Consider the system:

$$x_1 + 3x_2 - 2x_3 = 3$$
$$2x_1 + 6x_2 - 2x_3 + 4x_4 = 18$$
$$x_2 + x_3 + 3x_4 = 10$$

By elimination method,

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix} \xrightarrow[E_3(\frac{1}{2}),\, E_{23}(-1),\, E_{12}(-3)]{E_{21}(-2),\, E_{23}} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}.$$

Thus the solutions are: $x_1 = 3 - x_4$, $x_2 = 4 - x_4$ & $x_4 = 6 - 2x_4$,

ie $(3-t, 4-t, 6-2t, t)$, $t \in \mathbb{R}$.

Theorem: Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of a vector space V. If $\{w_1, w_2, \ldots, w_m\}$ is a set of vectors of V with $m > n$, then the set is L.D.

Proof (*): We will show that $\exists\ (\alpha_1, \alpha_2, \ldots, \alpha_m) \neq (0,0,\ldots 0)$ s.t

$$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m = 0$$

Since $\{v_i\}_1^n$ is a basis each w_i can be written as a linear combination of elements of $\{v_i : i=1,\dots n\}$. Thus

$$\alpha_1 w_1 + \dots + \alpha_m w_m = 0 \iff \alpha_1\Big(\sum_{j=1}^{n} a_{j1} v_j\Big) + \alpha_2\Big(\sum_{j=1}^{n} a_{j2} v_j\Big) + \dots + \alpha_m\Big(\sum_{j=1}^{n} a_{jm} v_j\Big) = 0$$

$$\iff \Big(\sum_{i=1}^{m} \alpha_i a_{1i}\Big) v_1 + \Big(\sum_{i=1}^{m} \alpha_i a_{2i}\Big) v_2 + \dots + \Big(\sum_{i=1}^{m} \alpha_i a_{ni}\Big) v_n = 0$$

$$\iff \sum_{i=1}^{m} \alpha_i a_{1i} = \dots = \sum_{i=1}^{m} \alpha_i a_{ni} = 0 \quad \left(\text{because } \{v_i\}_1^n \text{ is LI}\right).$$

There are n equations with m unknowns $\alpha_1, \alpha_2, \dots, \alpha_m$ & $m>n$ in the above equations. By the above theorem, the system has a non zero solution $(\alpha_1, \dots, \alpha_m)$. Hence $\{w_1, \dots, w_m\}$ is L.D. ▨

<u>Basis for vector space:</u> To be defined.

<u>Finite dimensional vector space:</u> A vector space V is said to be finite dimensional if there exists a basis consisting of finite number of elements. Otherwise, V is called infinite dimensional.

The following result is an immediate consequence of the previous result

<u>Theorem:</u> Let V be a finite dimensional vector space (f.d.v.s). Then any two bases of V have the same number of elements.

<u>Dimension:</u> The dimension of a f.d.v.s. V, denoted by $\dim(V)$, is the number of elements in a basis of V.

<u>Theorem:</u> Let $\dim(V) = n$. Then any L.I. set of n vectors of V is a basis of V

<u>Proof:</u> Let $\{x_1, x_2, \dots, x_n\}$ be L.I and $x \in V$ s.t. $x \neq x_i \ \forall i$. Then the set $\{x_1, x_2, \dots, x_n, x\}$ is L.D. Therefore, $\exists$ $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ and $\alpha \neq 0$ s.t $\sum_{i=1}^{n} \alpha_i x_i + \alpha x = 0$.

This implies that $x = \sum_{i=1}^{n} \frac{\alpha_i}{-\alpha} x_i$, i.e., span $\{x_1, x_2, \dots x_n\} = V$.

Therefore $\{x_1, x_2, \dots, x_n\}$ is a basis. ▨

In this course we will only deal with f.d.v.s.

<u>Prob1</u>: Let $S = \{ x_1 = (1,1,1,1),\ x_2 = (1,1,-1,1),\ x_3 = (1,1,0,1),\ x_4 = (1,-1,1,1) \}$ be a subset of $\mathbb{R}^4$. Find a basis of span(S).

<u>Sol</u>: consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$. Apply row reduction to A, we get

$$A \xrightarrow[E_{41}(-1)]{E_{21}(-1),\, E_{31}(-1)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \xrightarrow{E_{23}(-2)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} = U$$

The three non zero row vectors of U are clearly L.I and they also span(S) because, each vector x_i, $i = 1,2,3,4$ can be expressed as a linear combination of these three non zero row vectors of U.

Therefore, $\{ (1,1,1,1), (0,0,-1,0), (0,-2,0,0) \}$ is a basis of $L(S)$. ▨

<u>Problem 2</u>: Let $S = \{ x_1 = (1,-2,5,-3),\ x_2 = (0,1,1,4),\ x_3 = (1,0,1,0) \}$. Find a basis for span(S) and extend it to a basis of $\mathbb{R}^4$.

<u>Sol</u>: Note that $\dim(\text{span}(S)) \le 3$. consider $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Apply row reduction to A, we get

$$A \xrightarrow[E_{31}(2)]{E_{31}(-1)} \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -6 & -5 \end{bmatrix} = V.$$

Note that the set of row vectors of V are L.I. and hence it is a basis for span(S).

To extend it to a basis of $\mathbb{R}^4$, add a vector of the form $(0,0,0,t)$, $t \neq 0$, for example $(0,0,0,1)$ to the set.

Here note that $(0,0,0,1) \in \mathbb{R}^4 \setminus \text{span}(S)$. Hence the set $\{ x_1, x_2, x_3;\ (0,0,0,1) \}$ is L.I. Since the $\dim(\mathbb{R}^4) = 4$, the set becomes a basis by previous theorem. ▨

In fact any L.I subset of a f.d.v.s can be extended to a basis as did above.