Lecture 9

In the previous lecture we had seen that any L.I subset of a f.d.v.space V can be extended to a basis by adding more vectors. Let us repeat the proof of this fact. Suppose S is a L.I subset of V. If $L(S) = V$, then S is a basis of V. If not, choose $x \in V \setminus L(S)$. Then the set $\{S, x\}$ is LI (why?). If $L(\{S, x\}) = V$, $\{S, x\}$ is a basis. Otherwise continue the process.

Let us see one more example.

Example: Let $V = \{(v, w, x, y, z) \in \mathbb{R}^5 : v + x - 3y + z = 0\}$

$W = \{(v, w, x, y, z) \in \mathbb{R}^5 : w - x - z = 0 \text{ and } v = y\}$

Find bases of V & W containing the basis of $V \cap W$.

(Note that if V & V are linear spaces then $U \cap V$ is a linear space)

Solution: First let us find a basis of $V \cap W$. Note that

$$V \cap W = \{(v, w, x, y, z) \in \mathbb{R}^5 : v + x - 3y + z = 0,\ w - x - z = 0 \ \& \ v = y\}$$

$$= \{(v, w, x, y, z) \in \mathbb{R}^5 : v = y,\ z = 2y - x \text{ and } w = 2y\}.$$

$$= \{(y, 2y, x, y, 2y - x) \in \mathbb{R}^5 : x, y \in \mathbb{R}\}$$

$$= \{y(1, 2, 0, 1, 2) + x(0, 0, 1, 0, -1) : x, y \in \mathbb{R}\}$$

$$= \text{span}\{(1, 2, 0, 1, 2), (0, 0, 1, 0, -1)\}.$$

Hence the set $\{(1, 2, 0, 1, 2), (0, 0, 1, 0, -1)\}$ is a basis of $V \cap W$ because it is L.I.

Note that $W = \{(y, x + z, x, y, z) : x, y, z \in \mathbb{R}\}$

$$= \{x(0,1,1,0,0) + y(1,0,0,1,0) + z(0,1,0,0,1) : x, y, z \in \mathbb{R}\}$$

$$= \text{span}\{(0,1,1,0,0), (1,0,0,1,0), (0,1,0,0,1)\}.$$

Therefore $\dim(W) \leq 3$. In fact the $\dim(W) = 3$.

To extend the basis of $V\cap W$ to a basis of W, choose one element in $W\setminus(V\cap W)$, for example, choose $(1,1,1,1,0)\in W\setminus(V\cap W)$. Verify that $\{(1,2,0,1,2),(0,0,1,0,-1),(1,1,1,1,0)\}$ is a L.I subset of W. Since $\dim(W)\leq 3$, this set has be a basis of W.

Note that $V=\{(v,w,x,y,3y-v-x) : x,y,v,w\in\mathbb{R}^4\}$

$$=\{v(1,0,0,0,-1)+w(0,1,0,0,0)+x(0,0,1,0,-1)+y(0,0,0,1,3) : x,y,v,w\in\mathbb{R}^4\}$$

Therefore $\dim(V)\leq 4$.

Verify that $\{(1,2,0,1,2),(0,0,1,0,-1),(0,1,0,0,0),(1,1,0,0,-1)\}$ is a L.I. subset of V; therefore, it is a basis of V. ▨

It is easy to verify that if W_1 and W_2 are subspaces of a vector space V, then W_1+W_2 is a subspace of V and W_1 & W_2 are subspaces of W_1+W_2. For example, if W_1 and W_2 are one dimensional subspace of $\mathbb{R}^3$ then W_1+W_2 is the plane containing W_1 and W_2 when $W_1\neq W_2$ and it is W_1 if $W_1=W_2$. If $W_1\neq W_2$, in this case, we see that

$$2=\dim(W_1+W_2)=(\dim W_1=1)+(\dim W_2=1)+(\dim(W_1\cap W_2)=0).$$

If $W_1=W_2$, then

$$1=\dim(W_1+W_2)=\dim W_1+\dim W_2-\dim(W_1\cap W_2)=1+1-1.$$

In general, we have the following.

Theorem: If W_1 & W_2 are subspaces of a vector space V, then

$$\dim(W_1+W_2)=\dim W_1+\dim W_2-\dim(W_1\cap W_2).$$

We will not present the proof here; however we will use this result.

Example: Let $W_1=\operatorname{span}\{(1,3,-2,2,3),(0,1,-1,2,-1),(0,0,0,0,1)\}$

& $W_2=\operatorname{span}\{(1,3,0,2,1),(1,5,-6,6,3),(1,2,3,0,0)\}$.

Find a basis for W_1+W_2, $\dim(W_1\cap W_2)$ & basis for $W_1\cap W_2$.

<u>Solution:</u> It is easy to verify that the set

$$\{(1,3,-2,2,3),\ (0,1,-1,2,-1),\ (0,0,0,0,1)\}$$

is L.I (and hence it is a L.I. subset of W_1+W_2). Therefore $\dim(W_1)=3$. Let us see the $\dim(W_2)$. Taking the vectors as rows of a matrix & applying row operations we see that.

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 1 & 2 & 3 & 0 & 0 \end{bmatrix} \xrightarrow[R_3+\frac{1}{2}(R_2-2R_1)]{R_2-R_1} \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $\dim W_2 = 2$. By previous theorem $\dim(W_1+W_2)\leq 5$.

Note that $W_2 = \text{span}\{(1,3,0,2,1),\ (0,2,-6,4,2)\}$ &

$W_1+W_2 = \text{span}\{(1,3,-2,2,3),(0,1,-1,2,-1),(0,0,0,0,1),(1,3,0,2,1),(0,2,-6,4,2)\}$ (why?)

To find a basis of W_1+W_2, use elimination method:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \end{bmatrix} \xrightarrow[(R_4-R_1)(1/2)]{(R_5-2R_2)(-\frac{1}{4})} \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This shows that

$W_1+W_2 = \text{span}\{S\}$, where

$$S=\{(1,3,-2,2,3),(0,1,-1,2,-1),(0,0,1,0,-1),(0,0,0,0,1)\}$$

Since the set S is L.I, it is a basis for W_1+W_2.

observe that $\dim(W_1\cap W_2)=1$, because of the previous theorem.

<u>To find a basis for $W_1\cap W_2$:</u>

Let $x=(x_1,x_2,x_3,x_4,x_5)\in W_1\cap W_2$. Then consider the following matrix consisting of basis vectors of W_1 & x as rows:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \xrightarrow[R_4-(x_2-3x_1)R_2]{R_4-x_1R_1} \begin{bmatrix} 1 & 3 & -2 & 2 & \overbrace{3}^{-9(4)} \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -x_1+x_2+x_3 & 4x_1-2x_2+x_4 & 0 \end{bmatrix}$$

Since $\dim(W_1) = 3$, the last row vector has to the zero vector (otherwise, there will be four vectors in W_1 which are L. I). So we get two equations:

$$-x_1 + x_2 + x_3 = 0 \quad \text{----}(1)$$

$$4x_1 - 2x_2 + x_4 = 0 \quad \text{----}(2)$$

Similarly from W_2 we get the following equations:

$$-9x_1 + 3x_2 + x_3 = 0 \quad \text{---}(3)$$

$$4x_1 - 2x_2 + x_4 = 0$$

$$2x_1 - x_2 + x_5 = 0 \quad \text{----}(4)$$

There are four equations with five unknowns. From equation (1) and (3), we get that $-4x_1 + x_2 = 0$. By taking $x_1 = 1$, we get a solution $(1, 4, -3, 4, 2)$. Therefore,

$$W_1 \cap W_2 = \text{span}\{(1, 4, -3, 4, 2)\} = \{t(1, 4, -3, 4, 2) : t \in \mathbb{R}\}. \blacksquare$$