LECTURE 1:COMPLEX NUMBERS AND COMPLEX DIFFERENTIATION

1. Introduction

The study of complex numbers began to find roots of the polynomial equation $x^2+1=0$. It turns out that this equation does not have any real root. Thus one needs to go out of real numbers to get a solution of the above equation. Mathematicians also realized quite early that the study of complex numbers can lead to evaluations of certain integrals which are difficult to solve otherwise. For example, using complex numbers one can prove that

$$\int_{-\infty}^{\infty} \frac{e^{x/2}}{1+e^x} dx = \pi.$$

To do all these, it turns out that one needs also to study functions defined on the set of complex numbers. That is, we need to do calculus of functions (differential calculus and integral calculus) defined on the set of complex numbers.

Complex numbers : A complex number denoted by z is an ordered pair (x, y) with $x \in \mathbb{R}$, $y \in \mathbb{R}$. Where x is called real part of z and y is called the imaginary part of z. In symbol $x = \text{Re } z$,$y = \text{Im } z$. We denote $i = (0, 1)$ and hence $z = x + iy$ where the element x is identified with $(x, 0)$. By $\mathbb{C}$ we denote the set of all complex numbers, that is, $\mathbb{C} = \{z : z = x + iy, x \in \mathbb{R}, y \in \mathbb{R}\}$.

Addition and subtraction of complex numbers is defined exactly as in $\mathbb{R}^2$, for example, if

$z = x + iy$ and $z_1 = x_1 + iy_1$ then we define $z + z_1 = (x + x_1) + i(y + y_1)$.

Multiplication of complex numbers is something which makes it different from $\mathbb{R}^2$. let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers then we define

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Since $i = (0, 1)$ it follows from above that $i^2 = -1$.

We can define division of complex numbers also. If $z \neq 0$ then we define $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$. From this we get

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{{x_2}^2 + {y_2}^2}.$$

The following are easy to check directly from definitions:

(1) $z_1 + z_2 = z_2 + z_1$.
(2) $z_1 z_2 = z_2 z_1$.
(3) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

There is another interesting operation on the set of complex numbers called conjugation.
If $z = x + iy$ is a complex number then its conjugate is defined by $\bar{z} = x - iy$. Conjugation has the following properties which follows easily from the definition:

(1) Re $z = \frac{1}{2}(z + \bar{z})$ and Im $z = \frac{1}{2i}(z - \bar{z})$.
(2) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
(3) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (note that it follows from this that if $\alpha \in \mathbb{R}$ then $\overline{\alpha z} = \alpha \bar{z}$).

Polar form of complex numbers : Let $z = x + iy$ be a complex number with x and y both nonzero. Then there exist unique $r \in (0, \infty)$, and $\theta \in (-\pi, \pi]$ such that $z = re^{i\theta}$ where $e^{i\theta} = \cos\theta + i \sin\theta$. r and θ are related to z by the relations $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$, that is, r is the distance of z from the origin and θ is the angle between z and the positive direction of the $X-$axis. θ is called the principal argument of z and is usually written as $\theta =$ Arg z. The reason to restrict θ in $(-\pi, \pi]$ is to get the uniqueness of representation (because a 2π rotation will not change the point after all). Thus Arg $z =$ arg $z + 2k\pi$ So, if θ is argument of z then so is $\theta + 2k\pi$. For example, arg $i = 2k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$.

Multiplication and division of complex numbers can also be represented in the polar form: Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ then :
$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$, (upto a multiple of 2π) and as $|e^{i\theta}| = 1$ it follows that $|z_1 z_2| = |z_1||z_2|$. The above happens simply because $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$. Also arg $(z_1 z_2) =$ arg $z_1 +$ arg z_2 (upto multiple of 2π). Regarding multiplication of complex numbers written in polar form we have the following theorem:

De Moiver's formula:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

We put it into immediate use in the following *important* problem.

Problem: Given a nonzero complex number z_0 and a natural number $n \in \mathbb{N}$ find all distinct complex numbers w such that $z_0 = w^n$.

Solution: First note that if w satisfies the above then $|w| = |z_0|^{\frac{1}{n}}$. So, if $z_0 = |z_0|(\cos\theta + i\sin\theta)$ we try to find α such that

$$|z_0|(\cos\theta + i\sin\theta) = [|z_0|^{\frac{1}{n}}(\cos\alpha + i\sin\alpha)]^n.$$

It then follows from De Moiver's formula that $\cos\theta = \cos n\alpha$ and $\sin\theta = \sin n\alpha$, that is, $n\alpha = \theta + 2k\pi \Rightarrow \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}$. Now we notice that the distinct values of w is given by $|z_0|^{\frac{1}{n}}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n})$, for, $k = 0, 1, 2, \ldots, n-1$. That is, other values of k will give complex numbers which is already obtained.

Note that the above complex roots are not written in polar coordinates as $\frac{\theta+2k\pi}{n}$ may not belong to $(-\pi, \pi]$. Note that the n many complex numbers we got lie on the circle of radius $|z_0|^{\frac{1}{n}}$ about the origin and constitute the vertices of a regular polygon of n sides.

Example 1. *Let us find all complex numbers w such that $w^3 = -1$. By the above the solutions are* $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\pi+i\sin\pi = -1$, *and* $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3} = \cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3}$ *(polar representation). Thus the last one is the complex conjugate of the first one.*

Next we discuss the notion of convergent sequences and continuous functions.

For sequences what we do is replace modulus of real numbers just by modulus of complex numbers. Thus $z_n \to z$ as $n \to \infty$ if $|z_n - z| \to 0$ (just by writing down the definition of modulus it is not at all difficult to show that: *$z_n \to z$ as $n \to \infty$ iff Re $z_n \to$ Re z and Im $z_n \to$ Im z as $n \to \infty$*). Thus the definitions of Cauchy sequence, bounded sequence etc. are same as in the case of real numbers. Using the notion of convergent sequences we can easily define the notion of continuity for functions $f : \mathbb{C} \to \mathbb{C}$.

Now we come to the notion of differentiabilty of functions $f : \mathbb{C} \to \mathbb{C}$. The definition looks same as in the case of functions $f : \mathbb{R} \to \mathbb{R}$ but has some fundamental differences. We need some notation to start with. Given $z_0 \in \mathbb{C}$ and $r > 0$ we denote the ball of radius r around z_0 by $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$

Definition: Let $A \subset \mathbb{C}$, $B_r(z_0) \subset A$ and $f : A \to \mathbb{C}$. Then f is called differentiable at z_0 if the limit

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exist finitely.

We observe that in the above definition h varies over the set of complex numbers. As example let $f(z) = z^2$. Then $f(z+h) - f(z) = 2zh + h^2$ and hence the above limit is $2z$ which it should be (in analogy with the real case). Note that this calculation does not use the fact that h varies over the set of complex numbers in any special way.

To see what is really involved let us look at the function $g(z) = \overline{z}$. As

$$\frac{g(z+h) - g(z)}{h} = \frac{\overline{h}}{h},$$

we choose $h = \frac{1}{n}$ and $h = \frac{i}{n}$. Because of the conjugation in the numerator we get different limits, 1 and -1 as $n \to \infty$. Thus the function g is not differentiable anywhere in $\mathbb{C}$.