Lecture 10-12 Taylor series and zeros of analytic functions

We have seen that if f is analytic on the disc $\{z / |z-z_0|<R\}$ then f can be represented by the power series $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$ in an unique fashion as $a_n = f^n(z_0)/n!$.
The radius of convergence of the power series is R. Given a function it can become difficult to write down the Taylor series because coefficients involve infinitely many derivatives. Depending on the function sometime we can find the Taylor series using Taylor series of known functions.

Example: 1) Taylor series of $f(z) = z^5 \sin 2z$ can be written as (around $z_0 = 0$), $f(z) = z^5 \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{z^{2(n+3)}}{(2n+1)!}$.

2) $f(z) = e^z/(z+1)$. We can find the Taylor series around $z_0=0$ the following way

$$\frac{e^z}{1+z} = \left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1-z+z^2-z^3+\cdots\right) = 1+z^2/2 - z^3/3 + \cdots$$

3) To write Taylor series of $f(z) = \sin^3 z$ around $z_0=0$ we can use the identity $\sin^3 z = \frac{3}{4}\sin z - \frac{1}{4}\sin 4z$ and use the Taylor series of $\sin z$ and $\sin 4z$.

4) We want to write Taylor series of $f(z) = \mathrm{Log}\, z$ around $z_0 = 1$ ($z_0 = 0$ cannot be taken as Log is analytic on $\mathbb{C}\setminus(\mathbb{R}_-\cup\{0\})$) Here we can use differentiation formula. But there is another way of doing it. Suppose that $f(z) = \mathrm{Log}(z) = \sum_{n=0}^{\infty} a_n (z-1)^n$. Then by term by term differentiation we get $\frac{1}{z} = \sum_{n=1}^{\infty} n a_n (z-1)^{n-1}$. On the other hand we have from the geometric series that $\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{n=1}^{\infty} (-1)^{n-1}(z-1)^{n-1}$ on $\{z / |z-1|<1\}$. Thus by equating coefficients we get that $a_n = \frac{(-1)^{n-1}}{n}$, $n \geq 1$. Clearly $a_0 = f(1) = \mathrm{Log}(1) = 0$. So $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$. It then also follows that $\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$, for $\{z / |z|<1\}$.

Zeros of analytic functions:

Given a fn. f, a zero of f is a pt. z_0 s.t. $f(z_0)=0$. Define $Z_f=\{z\in\mathbb{C}: f(z)=0\}$. We will see that there is some similarity between the zero set (that is, Z_f) of a polynomial and that of an analytic fn. Any polynomial of degree n has at most n zeros and hence one can construct a ball around a zero where no other zero enters (how?), that is, zeros are isolated. An analytic function can have infinitely many zeros ($f(z)=\sin\pi z$) but they are still isolated (see next theorem).

Identity theorem (for a disc): Suppose that f is analytic on $D=\{z\in\mathbb{C} / |z-z_0|<r\}$ and $f(z_0)=0$. Then, either a) $f(z)=0 \;\forall\, z\in D$, or, b) $\exists\, \varepsilon>0$ s.t. $\forall\, z\in B_\varepsilon(z_0)\setminus\{z_0\}$, $f(z)\neq 0$.
Consequently, if $\exists\, \{z_n\}\subseteq D$, $z_n\neq z_0$ for infinitely many n, $f(z_n)=0\;\forall\, n$, $z_n\to z_0$ then $f(z)=0\;\forall\, z\in D$.

Proof: Let $f(z)=\sum_{n=1}^{\infty} a_n(z-z_0)^n$, $z\in D$. If $a_n=0\;\forall\, n$ then a) holds. If $\exists\, n$ s.t. $a_n\neq 0$ then get the smallest n (say n_0) s.t. $a_{n_0}\neq 0$. Then $f(z)=\sum_{n=n_0}^{\infty} a_n(z-z_0)^n$ and hence

$$f(z)=(z-z_0)^{n_0}\left(a_{n_0}+a_{n_0+1}(z-z_0)+a_{n_0+2}(z-z_0)^2+\cdots\right)=(z-z_0)^{n_0}g(z)$$

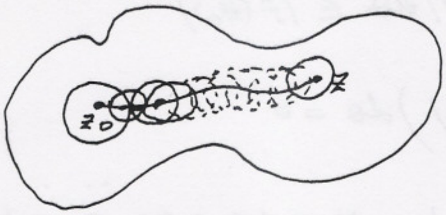
where g is analytic on D and $g(z_0)=a_{n_0}\neq 0$. As g is cont. $\exists\, \varepsilon>0$ s.t. $g(z)\neq 0\;\forall\, z\in B_\varepsilon(z_0) \Rightarrow f(z)\neq 0\;\forall\, z\in B_\varepsilon(z_0)\setminus\{z_0\}$.

Identity theorem (general form):

Let G be a domain and f be analytic on G. If $\exists\, \{z_n\}\subseteq G$, $z_n\neq z_0$, $f(z_n)=0\;\forall\, n$, $z_n\to z_0\in G$ then $f(z)=0\;\forall\, z\in G$.

Proof: (Sketch) Let z_0 be a limit pt. of Z_f in G (i.e. $\exists$ a seqn. $\{z_n\}\subseteq Z_f$ s.t. $z_n\to z_0$) then $f(z)=0\;\forall\, z\in B_r(z_0)\subseteq G$ for some $r>0$. Now let $z\in G$ arbitrary. Join z_0 & z by a path & cover the line segment by balls of radius

$r_0 < r$ as shown in the figure in such a way that center of a ball lies in the previous ball and apply the identity theorem of disc (which is a ball)



Corollary (Uniqueness theorem) Let f, g be analytic on a domain G s.t. $f(z_n) = g(z_n)$, $z_n \to z_0 \in G$ and $z_n \neq z_0$ for infinitely many n then $f(z) = g(z)$ $\forall z \in G$.

Pf: Apply the previous theorem on $F(z) = f(z) - g(z)$.

Applications: 1) Since $\sin^2 x + \cos^2 x = 1$ $\forall x \in \mathbb{R}$ we have $\sin^2 z + \cos^2 z = 1$ $\forall z \in \mathbb{C}$ ($G = \mathbb{C}$ & every pt. of $\mathbb{R}$ is a limit of a sequence in $\mathbb{C}$).

2) Let f be analytic on $B_1(0)$ as $f(x) = 0$ $\forall x \in (0,1)$ then $f(z) = 0$ $\forall z \in B_1(0)$. 3) Let f be entire and $f(\frac{1}{n}) = \sin\frac{1}{n}$ $\forall n \in \mathbb{N} \setminus \{0\}$ then $f(z) = \sin z$ $\forall z \in \mathbb{C}$.

4) $\nexists$ any analytic fn. f on $B_1(0)$ s.t. $f(x) = |x|^3$ on $(-1,1)$. As, $f(x) = x^3$ on $(0,1)$ we have $f(z) = z^3$ and since $f(x) = -x^3$ for $x \in (-1,0)$ we get $f(z) = -z^3$ – contradiction.

Note that in the real case the zeros are not isolated for a differentiable function (for example $f(x) = x^2 \sin\frac{1}{x}$, $x \in \mathbb{R}$).

We now give another application of the uniqueness theorem.

Maximum modulus principle:

Let f be a nonconstant analytic fn. on a domain G. Then $|f|$ does not attain a local maximum in G.

Proof: Suppose that $\exists$ $z_0 \in G$ s.t. $|f(z)| \leq |f(z_0)|$ $\forall z \in B_R(z_0)$ for some $R > 0$ s.t. $B_R(z_0) \subseteq G$. Now, by the mean value

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property we have $f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta \;\forall\; 0<r<R$. Thus

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \le |f(z_0)|$$

$$\Rightarrow \int_0^{2\pi}\left(|f(z_0)|-|f(z_0+re^{i\theta})|\right)d\theta = 0$$

$$\Rightarrow |f(z_0)| = |f(z_0+re^{i\theta})|,\; \forall\; 0\le\theta\le 2\pi \;\&\; 0<r<R$$

$\Rightarrow |f|$ is const. in $B_R(z_0)$

$\Rightarrow f$ is const. on $B_R(z_0)$ (by an assignment problem on CR equations).

$\Rightarrow f$ is const. on G by uniqueness theorem.

Corollary: 1) If f is analytic inside and on a simple closed curve c then $|f|$ attains its maximum only on the boundary.

2) If f is a nonconstant analytic fn. on a domain G & $f(z)\ne 0 \;\forall\; z\in D$ then $|f|$ cannot attain its minimum inside G (consider $g=\frac{1}{f}$).

Laurent Series

We have seen the advantages of expressing a function in Taylor series around a pt. z_0. If a function is analytic on a set $B_\varepsilon(z_0)\setminus\{z_0\}$ then we cannot expect to write the fn. in the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ because then the fn. will be analytic at z_0. However we can try to write a series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$ (for example $f(z)=\frac{1}{z}$ for $z\in B_1(0)\setminus\{0\}$).

A series of the above kind is called a Laurent series.

Example: Let $f(z)=\frac{1}{1-z}$, $z\ne 1$. On the set $\{z \mid |z|<1\}$ the fn. is analytic & has Taylor series $\sum_{n=0}^{\infty} z^n$.

Now consider the set $\{z \in \mathbb{C} \,/\, |z|>1\}$. The fn. is analytic here also but we cannot express it in the form $\sum a_n z^n$, because then it will also be valid at $z=1$ and will be analytic there. However we can get a series in the powers of $\frac{1}{z}$ as follows:

$$\frac{1}{1-z} = \frac{-1}{z(1-1/z)} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\cdots$$

which converges for $|z|>1$.

The next theorem shows that under some condition on f and the domain we can get Laurent series of f.

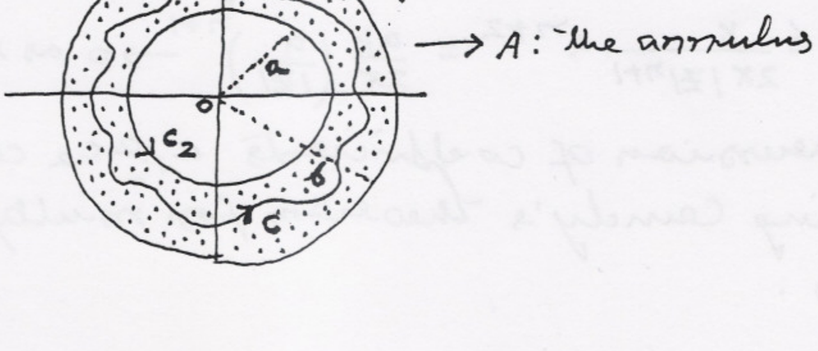
<u>Theorem</u>: Let $A = \{z \in \mathbb{C} : a < |z-z_0| < b\}$, $(0<a<b<\infty)$ and f be analytic on A and on the circles $\{z \,/\, |z-z_0| = a\}$, $\{z \,/\, |z-z_0|=b\}$. If c is any simple closed curve in A (enclosing the inner circle) then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

$$\left(\text{or},\ f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n\right)$$

where $a_n = \frac{1}{2\pi i}\int_c \frac{f(\omega)}{(\omega-z_0)^{n+1}}d\omega$, $b_n = \frac{1}{2\pi i}\int_c f(\omega)(\omega-z_0)^{n-1}d\omega$

$$\left(\text{or},\ c_n = \frac{1}{2\pi i}\int_c \frac{f(\omega)}{(\omega-z_0)^{n+1}}d\omega,\ \underline{n\in\mathbb{Z}}\right).$$

<u>Proof</u>: (*) Consider the following picture with $z_0=0$.



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By CIF for multiply connected domain, for $z \in A$ we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw$$

Consider the first integral and use the proof of Taylor's theorem to write

$$\frac{1}{2\pi i}\int_{C} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w^{n+1}}\,dw$$

using the series $\frac{1}{w-z} = \frac{1}{w(1-\frac{z}{w})} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$ (for this series to converge we need $|z|<|w|$, which we have as $w \in C_1$). Now the second integral, where the situation is $|z|>|w|$ as $w \in C_2$. We use

$$\frac{1}{w-z} = \frac{-1}{z(1-\frac{w}{z})} = -\left[\frac{1}{z} + \frac{w}{z^2} + \cdots + \frac{w^n}{z^{n+1}}\right] - \frac{1}{z-w}\left(\frac{w}{z}\right)^{n+1}$$

(using $\frac{1}{1-q} = 1+q+\cdots+q^n + \frac{q^{n+1}}{1-q}$).

$$\Rightarrow -\frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\left[\frac{1}{z}\int_{C_2} f(w)\,dw + \frac{1}{z^2}\int_{C_2} f(w)\,w\,dw + \cdots\cdots \right.$$

$$\left. + \frac{1}{z^{n+1}}\int_{C_2} f(w)\,w^n\,dw\right] + R_n(z)$$

where $R_n(z) = \frac{1}{2\pi i\, z^{n+1}}\int_{C_2} \frac{w^{n+1}}{z-w} f(w)\,dw$.

As $\frac{f(w)}{z-w}$ is bdd. on C_2 as a function of w (z is fixed) by K (say) we get from ML inequality that

$$|R_n(z)| \le \frac{K}{2\pi |z|^{n+1}}\, a^{n+2} = \frac{aK}{2\pi}\left(\frac{a}{|z|}\right)^{n+1} \to 0 \text{ as } n \to \infty.$$

In the expression of coefficients C_1 & C_2 can be replaced by C using Cauchy's theorem for multiply connected domains.

<u>Note</u>: Let f be analytic on an annulus $A = \{z / a < |z - z_0| < b\}$ then $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$ where c_n is given above.

The Laurent series of f on A is unique (proof will not be given)

<u>Example</u> Compute the Laurent series of $f(z) = \dfrac{1}{z(1-z)^2}$ on the sets $A_1 = \{z : 0 < |z-1| < 1\}$, $A_2 = \{z : |z-1| > 1\}$.

<u>For $z \in A_1$</u> : $f(z) = \dfrac{1}{(1-z)^2}\left[\dfrac{1}{1+(z-1)}\right]$

$$= \frac{1}{(1-z)^2} \sum_{n=0}^{\infty} (-1)^n (z-1)^n = \sum_{n=-2}^{\infty} (-1)^n (z-1)^n$$

<u>For $z \in A_2$</u> : $f(z) = \dfrac{1}{(z-1)^3}\left[\dfrac{1}{1+\frac{1}{z-1}}\right] = \displaystyle\sum_{n=-\infty}^{-3} (-1)^{n+1} (z-1)^n$

<u>Pole and residue</u>

<u>Defn.</u> (Pole) A pt. z_0 is called an isolated singular point of a fn. f if f fails to be analytic at z_0 but is analytic on $B_\varepsilon(z_0) \setminus \{z_0\}$ for some $\varepsilon > 0$. For such a z_0 $\exists$ a unique Laurent expansion of f: $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$.

The pt. z_0 is called a pole of order $m \geq 1$ if $c_{-m} \neq 0$ and $c_n = 0$ $\forall$ $n < -m$. If $m = 1$ then z_0 is called a simple pole.

<u>Example</u>: 1) $f(z) = \frac{1}{z}$ has a simple pole at $z = 0$

2) If f is analytic and $f(z_0) \neq 0$ then $g(z) = \dfrac{f(z)}{(z - z_0)^m}$ has a pole of order m at z_0 (follows by writing down the Laurent series of g using Taylor's

series of f: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Theorem z_0 is a pole of order m of f iff $\exists\, h$ s.t. $f(z) = \frac{h(z)}{(z-z_0)^m}$ with h analytic at z_0 and $h(z_0) \neq 0$.

Pf: If z_0 is a pole of order m for f then using the Laurent expansion of f we get

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + c_0 + c_1(z-z_0) + \cdots$$

$$= \frac{1}{(z-z_0)^m}\left[c_{-m} + c_{-m+1}(z-z_0) + \cdots\right]$$

$$= \frac{h(z)}{(z-z_0)^m}$$

$\therefore\ h(z_0) = c_{-m} \neq 0$ and h is analytic at z_0 as it has a convergent-Taylor series at z_0.

Example 1) As $\sin z = z\,h(z)$ with h analytic and $h(0) \neq 0$ we see that 0 is not a singularity for $\frac{\sin z}{z}$.

2) Let $f(z) = \frac{\pi \cot \pi z}{z^2} = \frac{\pi \cos \pi z}{z^2 \sin \pi z} = \frac{\pi}{z^3} \cdot \frac{\cos \pi z}{h(z)}$

$\Rightarrow 0$ is a pole of order 3 for f.

Theorem (L'Hospital rule)

Suppose f, g are analytic at z_0. If $f(z_0) = 0 = g(z_0)$ but $g'(z_0) \neq 0$, then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

Pf: As g is analytic so is g' & hence g' is cont. As $g'(z_0) \neq 0$ $\exists\, \varepsilon$ s.t. $\forall z \in B_\varepsilon(z_0)$, $g'(z) \neq 0$. As f & g are analytic we can get $\varepsilon' > 0$ s.t $f(z) \neq 0 \; \forall \; z \in B_{\varepsilon'}(z_0) \setminus \{z_0\}$ (zeros are isolated). Same is true for g. Now

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{[f(z) - f(z_0)]/z - z_0}{[g(z) - g(z_0)]/z - z_0} = \frac{f'(z_0)}{g'(z_0)}.$$

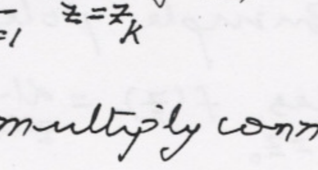
Defn. (Residue) The complex no. c_{-1}, which is the coefficient of $\frac{1}{z - z_0}$ in the Laurent expansion of f is called the residue of f at z_0 (Notation for residue of f at z_0 is $\operatorname{Res}_{z = z_0} f(z)$).

The reason for looking at residue comes from the Laurent expansion. We know that c_{-1} is given by the integral $\int_\gamma f(z)\,dz$. So without computing the Laurent series if we know the residue then we know the value of the integral $\int_\gamma f(z)\,dz$.

Cauchy's Residue Theorem:

Let f be analytic inside and on a simple closed contour C except for a finite no. of points $a_1, a_2, \ldots, a_N$ then $\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z = z_k} f(z)$.

Proof: By Cauchy's theorem for a multiply connected domain we have



$$\int_C f(z)\,dz - \sum_{j=1}^{N} \int_{c_j} f(z)\,dz = 0$$

$$\Rightarrow \int_C f(z)\,dz - 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{z = a_j} f(z) = 0.$$

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Now we need to have formulas to calculate the residues.

1) Residue at a simple pole:
Let f be analytic on $0<|z-z_0|<r$ and suppose that f has a simple pole at z_0 then $\operatorname{Res}_{z=z_0} f(z)=\lim_{z\to z_0}(z-z_0)f(z)$.

Pf: $f(z)=\sum_{n=-1}^{\infty} c_n(z-z_0)^n \Rightarrow \lim_{z\to z_0}(z-z_0)f(z)=c_{-1}$.

Cor: a) Let $f(z)=\dfrac{g(z)}{z-z_0}$ where g is analytic on $|z-z_0|<r$ and $g(z_0)\neq 0$ then $\operatorname{Res}_{z=z_0} f(z)=\lim_{z\to z_0} g(z)=g(z_0)$
(we can also prove this by Taylor expansion of g).

b) Let $f(z)=\dfrac{h(z)}{g(z)}$ where h, g are analytic on $|z-z_0|<r$, $h(z_0)\neq 0$, $g(z_0)=0$ and $g'(z_0)\neq 0$ then $\operatorname{Res}_{z=z_0} f(z)=\dfrac{h(z_0)}{g'(z_0)}$.

Proof: From the given condition $g(z)=(z-z_0)h_1(z)$ with $h_1(z_0)=\lim_{z\to z_0}\dfrac{g(z)}{z-z_0}=g'(z_0)\neq 0$. Thus
$f(z)=\dfrac{1}{z-z_0}\cdot\dfrac{h(z)}{h_1(z)}$ with $\dfrac{h(z_0)}{h_1(z_0)}\neq 0$. So z_0 is a simple pole of f. Hence

$$\operatorname{Res}_{z=z_0} f(z)=\lim_{z\to z_0}(z-z_0)\frac{h(z)}{g(z)}=\frac{h(z_0)}{g'(z_0)}.$$

2) Residue at a pole of order m:

Theorem: If f has a pole of order m at z_0 then

$$\operatorname{Res}_{z=z_0} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z).$$

Proof: Since f has a pole of order m at $z=z_0$

it follows from the Laurent-series of f that

$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \cdots + c_0 (z-z_0)^m + \cdots$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = (m-1)!\, c_{-1} + \text{terms involving } (z-z_0)$$

$$\Rightarrow \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = c_{-1}.$$

Example 1) Let $f(z) = \dfrac{1}{(z-2)(z^2+4)}$. Then f has simple poles at $2, \pm 2i$.

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} \frac{1}{z^2+4} = \frac{1}{8}$$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i} \frac{1}{(z-2)(z+2i)} = \frac{1}{4i(2i-2)}$$

$$\operatorname*{Res}_{z=-2i} f(z) = \frac{1}{4i(2i+2)}$$

2) $f(z) = \cot z$. Then f has a simple pole at $z=0$ and

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{z}{\sin z}\cdot \cos z = \lim_{z\to 0} \frac{1}{\cos z} \cos z = 1.$$