LECTURE 2: COMPLEX DIFFERENTIATION AND CAUCHY RIEMANN EQUATIONS

We have seen in the first lecture that the complex derivative of a function f at a point z_0 is defined as the limit

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h},$$

whenever the limit exist. We have also seen two examples *i)* if $f(z) = z^2$ then $f'(z) = 2z$, *ii)* the function $f(z) = \overline{z}$ is not a differentiable function. Now we will go for a detail study.

Theorem 1. *If f is differentiable at z_0 then f is continuous at z_0.*

Proof. Since $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ it follows that

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}(z-z_0) + f(z_0) = f(z_0).$$

□

The following results about derivatives follow exactly as in the case of reals:

(1) Derivative of a constant function is zero and $\frac{d}{dz}(z^n) = nz^{n-1}$, $n \in \mathbb{N}$.
(2) If $\alpha, \beta \in \mathbb{C}$ then $(\alpha f + \beta g)' = \alpha f' + \beta g'$.
(3) (**Chain Rule**) $\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)$ whenever all the terms make sense.

So much for similarity. To see the difference of complex derivatives and the derivatives of functions of two real variables we look at the following example.

Example 2. *Consider the function $f : \mathbb{C} \to \mathbb{R}$ given by $f(z) = |z|^2$. Since $z = x + iy$ the function f can also be thought of as a function from $\mathbb{R}^2$ to $\mathbb{R}$. From this point of view the function f can also be written as $f(x,y) = x^2 + y^2$. Since the partial derivatives of f are continuous throughout $\mathbb{R}^2$ it follows that f is differentiable everywhere on $\mathbb{R}^2$. But what happens if we now view f as a function on $\mathbb{C}$ and think about complex differentiability? it is clear that f is differentiable at zero as*

$$\lim_{h\to 0} \frac{|h|^2}{h} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^2+h_2^2}{h_1+ih_2} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^2+h_2^2}{h_1^2+h_2^2}(h_1 - ih_2) = 0$$

(we have used the important fact that $|z|^2 = z\overline{z}$). On the other hand $\lim_{h\to 0} \frac{|z+h|^2-|z|^2}{h} = \lim_{h\to 0} \frac{z\bar{h}+\bar{z}h}{h}$ does not exist for $z \neq 0$ as $\lim_{h\to 0} \frac{\overline{h}}{h}$ does not exist.

So we need to find a necessary condition for differentiability of a function of a complex variable z. These are called Cauchy- Riemann equations (*CR equation* for short) given in the following theorem.

We need the following notation to express the theorem which deals with the real part and imaginary part of a function of a complex variable. Let $f : \mathbb{C} \to \mathbb{C}$ be a function then
$f(z) = f(x, y) = u(x, y) + iv(x, y)$. The functions u and v can be thought of as real valued functions defined on subsets of $\mathbb{R}^2$ and are called real and imaginary part of f respectively (u=Re f, v=Im f).

Theorem 3. *Suppose that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at the point $z_0 = (x_0, y_0)$ and*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Thus equating the real and imaginary parts we get

$$u_x = v_y,\ u_y = -v_x,\ \text{at } z_0 = x_0 + iy_0 \qquad (\textbf{Cauchy Riemann equations}).$$

Proof. Since f is differentiable at z_0 we have by varying h over the set of real numbers

$$\begin{aligned} f'(z_0) = f'(x_0 + iy_0) &= \lim_{h\to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0) + i[v(x_0 + h, y_0) - v(x_0, y_0]}{h} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0), \end{aligned}$$

and

$$\begin{aligned} f'(z_0) &= f'(x_0 + iy_0) \\ &= \lim_{h\to 0} \frac{u(x_0, y_0 + h) - u(x_0, y_0) + i[v(x_0, y_0 + h) - v(x_0, y_0)]}{ih} \\ &= \lim_{h\to 0} \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{h} - i \lim_{h\to 0} \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h} \qquad (\text{as } \frac{1}{i} = -i) \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0). \end{aligned}$$

Now we can just compare the real and imaginary parts of $f'(z_0)$. This completes the proof. □

Note that the crux of the proof is to approach the point $(0, 0)$ through real axis (X-axis) and through imaginary axis (Y-axis) (it is the same way we have shown that the function $f(z) = \overline{z}$ is not differentiable).

Now, CR equations has some magical consequences, some of which is mentioned below.

(1) If $f : \mathbb{C} \to \mathbb{C}$ is such that $f'(z) = 0$ for all $z \in \mathbb{C}$, then f is a constant function. This is because, by CR equation $u_x = u_y = v_x = v_y = 0$. So by MVT of two variable calculus u and v are constant function and hence so is f.
(2) If $f : \mathbb{C} \to \mathbb{C}$ is differentiable everywhere and $f(z)$ is real for all $z \in \mathbb{C}$ then f is a constant function. This follows from CR equation as $v(x, y) = 0$ for all $x + iy \in \mathbb{C}$ and hence all partial derivatives of v is also zero and hence the same is true for u. Thus the function $f(z) = |z|^2$ is not differentiable for $z \neq 0$.

However CR equations do not give a sufficient criteria for differentiability.

Example 4. *Let $f(z) = \overline{z}^2/z$, if $z \neq 0$ and $f(0) = 0$. It is easy to see that this function is not differentiable at* 0. *By definition*
$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{(\bar{h})^2}{h^2}.$$
Now by choosing h to be real we get the limit to be 1 *and replacing h by $h + ih$ we see that the limit is* -1. *But real and imaginary parts of f satisfies CR equations at* $z = 0$ *(check this!).*

If we add some more conditions on the partial derivatives of u and v along with CR equations then one can conclude that the function is differentiable. We state a theorem (without the proof) for the precise statement.

Theorem 5. *(Converse of CR relations) $f = u + iv$ be defined on $B_r(z_0)$ such that u_x, u_y, v_x, v_y exist on $B_r(z_0)$ and are continuous at z_0. If u and v satisfies CR equations then $f'(z_0)$ exist and $f' = u_x + iv_x$.*

Example 6. *Using the above result we can immediately check that the functions*

(1) $f(x + iy) = x^3 - 3xy^2 + i(3x^2y - y^3)$
(2) $f(x + iy) = e^{-y} \cos x + ie^{-y} \sin x$

are differentiable everywhere in the complex plane.

CR equations can also be expressed in the polar coordinates.
Exercise: Using $x = r \cos \theta$, $y = r \sin \theta$ and the chain rule $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ prove that the CR equation is equivalent to
$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta.$$

As in the cartesian case, it can be proved that if $u_r, u_\theta, v_r, v_\theta$ are continuous and satisfies CR equations then the function is differentiable.