LECTURE 3: ANALYTIC FUNCTIONS AND POWER SERIES

We are interested in a class of differentiable functions called *analytic functions.* But first let me explain the notion of *open sets.*

Definition 1. *A subset $U \subseteq \mathbb{C}$ is called open if for every $z \in \mathbb{C}$ there exist $r_z > 0$ such that the ball $B_{r_z}(z) \subset U$.*

For example, the sets $\mathbb{C}$, $\mathbb{C}^*$, $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : \text{Re } (z) > 0\}$ are open sets. But the sets $\{z \in \mathbb{C} : |z| \leq 1\}$ and $\{z \in \mathbb{C} : \text{Re } (z) \geq 0\}$ are not open sets as, for example, in the first case we cannot find balls with centers at $|z| = 1$ for any z which is contained in the set $\{z \in \mathbb{C} : |z| \leq 1\}$. Similar is the case for the second one.

Definition 2. *A function f is called analytic at a point $z_0 \in \mathbb{C}$ if there exist $r > 0$ such that f is differentiable at every point $z \in B_r(z_0)$.*

A function is called analytic in an open set $U \subseteq \mathbb{C}$ if it is analytic at each point U.

Example 3. *Here are some examples*

(1) *For $n \in \mathbb{N}$ and complex numbers $a_0, \ldots, a_n$ the polynomial $f(z) = \sum_{k=0}^{n} a_k z^k$ is an analytic function for all $z \in \mathbb{C}$.*
(2) *The function $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$. In fact, any rational function (functions of the form $\frac{p(z)}{q(z)}$ where p and q are polynomial functions) is analytic in their domain of definition.*
(3) *The function $f(z) = |z|^2$ is not analytic at any point (though it is differentiable at $z = 0$).*

But how does one produce a *large* class of examples of analytic functions? It turns out that they can be build out of polynomials, that is, they are actually given by *power series* (this is another difference with the reals: if $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere on $\mathbb{R}$ then it is not necessary that f is given by a power series). So we need to develop the notion of power series.

A series of complex numbers is an infinite sum of the form $\sum_{n=0}^{\infty} z_n$ where all the z_ns are complex numbers (for example, $\sum_{n=1}^{\infty} i^n$, $\sum_{n=1}^{\infty}(\frac{1}{n} + \frac{i}{2^n})$). The notion of Convergence is defined exactly as in the reals, that is, $\sum_{n=0}^{\infty} z_n$ converges if the sequence of partial sums $\{s_m = \sum_{n=0}^{m} z_n\}$ converges to some complex number. The following are simple consequences of the definition:

(1) If $\sum_{n=0}^{\infty} z_n$ converges then $z_n \to 0$ as $n \to \infty$ (as $z_{n+1} = s_{n+1} - s_n \to 0$ as $n \to \infty$).

(2) If $\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty}(x_n + iy_n)$ converges then $\sum_{n=0}^{\infty} x_n$, $\sum_{n=0}^{\infty} y_n$ converges and we have $\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} x_n + i\sum_{n=0}^{\infty} y_n$.

For example, $\sum_{n=1}^{\infty} \frac{(-1)^n+i}{n^2}$ converges but $\sum_{n=0}^{\infty}(1+i)^n$ does not converge as $|(1+i)^n| = (\sqrt{2})^n$ does not converge to zero as $n \to \infty$.

Theorem 4. *(Comparison test) If $|z_n| \leq M_n$ for $M_n \in \mathbb{R}$ and $\sum_{n=0}^{\infty} M_n$ converges then $\sum_{n=0}^{\infty} |z_n|$ converges and hence so does $\sum_{n=0}^{\infty} z_n$.*

Proof. It is clear that $|x_n| \leq |z_n| \leq M_n$ and $|y_n| \leq |z_n| \leq M_n$. So $\sum_{n=0}^{\infty} x_n$, $\sum_{n=0}^{\infty} y_n$ converges absolutely and hence convergent. So $\sum_{n=0}^{\infty} z_n$ converges.

□

Example 5. (1) *It follows from the above theorem that $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$ converges, as $|3+4i| = 5$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.*

(2) *The most fundamental series for us is the geometric series $\sum_{n=1}^{\infty} z^n$. By comparison with its real counterpart it follows that the above series converges for $|z| < 1$ (to $\frac{1}{1-z}$) and diverges for $|z| \geq 1$ (as $|z_n|$ does not converge to 0 as $n \to \infty$).*

We note that, like the series of real terms, we also have ratio test for the series of complex numbers, which says: *If $\lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|} = L$ and $L < 1$ then $\sum_{n=0}^{\infty} z_n$ converges.* Similarly, we also have analogue of the root test.

Definition 6. *A series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, where $a_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$ is called a power series around the point z_0..*

To develop the theory, as in the real case, we are going to assume that $z_0 = 0$. So, the geometric series, given in the previous example, is a power series. The following theorem shows that *a power series is very good when it is good.*

Theorem 7. *If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \in \mathbb{C}$ then it converges for all $z \in \mathbb{C}$ such that $|z| < |z_0|$ (which is a disc without the boundary around the origin with radius $|z_0|$.)*

Proof. It follows from the hypothesis that there exist $M \geq 0$ such that $|a_n z_0^n| \leq M$ for all $n \in \mathbb{N}$. The proof now follows from the comparison theorem, behaviour of geometric series and the observation

$$|a_n z^n| = |a_n z_0|^n |\frac{z}{z_0}|^n \leq M|\frac{z}{z_0}|^n.$$

□

Note that if the series $\sum_{n=0}^{\infty} a_n z_0^n$ diverges then so does the series $\sum_{n=0}^{\infty} a_n z^n$ for $|z| > |z_0$ by the previous result.

Definition 8. *(Radius of convergence) The radius of convergence of a power series* $\sum_{n=0}^{\infty} a_n z^n$ *is defined as*

$$R = sup \ \{|z| : \sum_{n=0}^{\infty} a_n z^n \ \textit{converges}\}.$$

It is clear that $R \geq 0$ (however it is possible that $R = 0$) and if $|z| < R$ (resp. $|z| > R$) then the power series converges (resp. diverges).

Example 9. *a) For* $\sum_{n=0}^{\infty} n! z^n$, $R = 0$. *b) For* $\sum_{n=0}^{\infty} z^n$, $R = 1$. *c) For* $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $R = 1$. *d) For* $\sum_{n=1}^{\infty} \frac{z^n}{n!}$, $R = \infty$.

Remark: Note that no conclusion about convergence can be drawn if $|z| = R$. The power series in c) above does not converge if $z = 1$ but converges if $z = -1$.

The formula for calculating R goes exacrly as in the case of reals, that is,

$$\frac{1}{R} = \lim_{n \to \infty} |a_n|^{\frac{1}{n}} = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|},$$

whenever the above limits exist (with the supposition that division by ∞ (resp. 0) produces 0 (resp. ∞)).

To proceed further we need the following lemma which says that if we perform term by term differentiation for a power series then the new power series has the same radius of convergence as the old one.

Lemma 10. *If* $\sum_{n=0}^{\infty} a_n z^n$ *has radius of convergence* $R > 0$ *then the series* $\sum_{n=1}^{\infty} n a_n z^{n-1}$ *converges for* $|z| < R$.

Proof. Let $|z| = r < R$. We will show that $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges. Choose s such that $r < s < R$. So $\sum_{n=0}^{\infty} a_n s^n$ converges and hence there exist $M > 0$ such that $|a_n| \leq \frac{M}{s^n}$ (why?). Thus

$$|n a_n z^{n-1}| \leq n \frac{M}{s^n} r^{n-1} = \frac{M}{r} \frac{n}{(s/r)^n}.$$

As $r < s$ it follows from root test that the series $\sum_{n=0}^{\infty} \frac{n}{(s/r)^n}$ converges (as $\lim_{n \to \infty} n^{1/n} = 1$). The proof now follows from the comparison test.

□