LECTURE 4: DERIVATIVE OF POWER SERIES AND COMPLEX EXPONENTIAL

The reason of dealing with power series is that they provide examples of analytic functions.

Theorem 1. *If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then the function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is differentiable on $S = \{z \in \mathbb{C} : |z| < R\}$, and the derivative is $f(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.*

Proof. $(*)$ We will show that $|\frac{F(z+h)-F(z)}{h} - f(z)| \to 0$ as $h \to 0$ (in $\mathbb{C}$), whenever $|z| < R$. Using the binomial theorem $(z+h)^n = \sum_{k=0}^{n} \binom{n}{k} h^k z^{n-k}$ we get

$$\begin{aligned}
\frac{F(z+h)-F(z)}{h} - f(z) &= \sum_{n=0}^{\infty} a_n \frac{(z+h)^n - z^n - hnz^{n-1}}{h} \\
&= \sum_{n=0}^{\infty} \frac{a_n}{h} (\sum_{k=2}^{n} \binom{n}{k} h^k z^{n-k}) \\
&= \sum_{n=0}^{\infty} a_n h (\sum_{k=2}^{n} \binom{n}{k} h^{k-2} z^{n-k}) \\
&= \sum_{n=0}^{\infty} a_n h (\sum_{j=0}^{n-2} \binom{n}{j+2} h^j z^{n-2-j}) \quad \text{(by putting } j = k-2\text{)}.
\end{aligned}$$

By using the easily verifiable fact that $\binom{n}{j+2} \leq n(n-1)\binom{n-2}{j}$, we obtain

$$\begin{aligned}
|\frac{F(z+h)-F(z)}{h} - f(z)| &\leq |h| \sum_{n=0}^{\infty} n(n-1)|a_n| (\sum_{j=0}^{n-2} \binom{n-2}{j} |h|^j |z|^{n-2-j}) \\
&= |h| \sum_{n=0}^{\infty} n(n-1)|a_n| (|z| + |h|)^{n-2}.
\end{aligned}$$

We already know that the series $\sum_{n=0}^{\infty} n(n-1)|a_n||z|^{n-2}$ converges for $|z| < R$. Now, for $|z| < R$ and $h \to 0$ we have $|z| + |h| < R$ eventually. It thus follows from above that $|\frac{F(z+h)-F(z)}{h} - f(z)| \to 0$ as $h \to 0$, whenever $|z| < R$. □

We are now going to define the complex analogue of the exponential function, that is, e^x.

Definition 2. *(Exponential function) We define* $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ *for all* $z \in \mathbb{C}$.

Since $|\frac{a_{n+1}}{a_n}| = \frac{1}{(n+1)} \to 0$, the series converges for all $z \in \mathbb{C}$. The following theorem summarizes important properties of the exponential.

Theorem 3. *The function* $f(z) = e^z$ *is analytic on* $\mathbb{C}$ *and satisfies the following properties*

$$i)\ \frac{d}{dz}(e^z) = e^z \ \ ii)\ e^{z_1+z_2} = e^{z_1}e^{z_2} \ \ iii)\ e^{i\theta} = \cos\theta + i\sin\theta,\ \ \theta \in \mathbb{R}.$$

Proof. By the previous result e^z is an analytic function on $\mathbb{C}$ and

$$\frac{d}{dz}(e^z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = e^z.$$

We define $g(z) = e^z e^{z_1+z_2-z}$. Then g is analytic on $\mathbb{C}$ and $g'(z) = 0$ for all $z \in \mathbb{C}$. It follows from CR equations that $g(z) = \alpha$ for some $\alpha \in \mathbb{C}$. Since $g(0) = \alpha = e^{z_1+z_2}$ we get that $e^z e^{z_1+z_2-z} = e^{z_1+z_2}$. By choosing $z = z_1$ the result follows.

For the last part we have,

$$\begin{aligned} e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} &= \sum_{n=0}^{\infty} [\frac{1}{(2n)!}(i\theta)^{2n} + \frac{1}{(2n+1)!}(i\theta)^{2n+1}] \\ &= \sum_{n=0}^{\infty} [\frac{\theta^{2n}(i^2)^n}{(2n)!} + i\frac{\theta^{2n+1}(i^2)^n}{(2n+1)!}] = \cos\theta + i\sin\theta. \end{aligned}$$

□

We have the following observations to make.

(1) Since $e^z e^{-z} = 1$ it follows that $e^z \neq 0$ for all $z \in \mathbb{C}$.
(2) $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$ and $\overline{e^z} = e^{\bar{z}} = e^x e^{-yi}$.
(3) $e^{z+2n\pi i} = e^x(\cos(2n\pi + y) + i\sin(2n\pi + y)) = e^x(\cos y + i\sin y) = e^z$. Thus complex exponential is a periodic function with period $2\pi i$ and hence it is *not injective*, unlike the real exponential.
(4) It follows now easily that $e^z = 1 \iff z = 2n\pi i$ for some $n \in \mathbb{Z}$ and hence $e^{z_1} = e^{z_2} \iff z_2 = z_1 + 2n\pi i$, for some $n \in \mathbb{Z}$.

Surjectivity of Exponential: We know that complex exponential is not an injective function however it is surjective from $\mathbb{C}$ to $\mathbb{C}\setminus\{0\}$. If $w \in \mathbb{C}\setminus\{0\}$ then using polar coordinates we can write $w = |w|e^{i\theta}$ where $\theta \in (-\pi, \pi]$. If we define $z = \log|w| + i\theta$ then $e^z = e^{\log|w|+i\theta} = e^{\log|w|}e^{i\theta} = w$. In fact, by our previous observation, it follows that $e^{\log|w|+i(\theta+2n\pi i)} = w$ (not surprising as exponential is not injective). Thus we have that $e^{\log|w|+i\text{Arg}w} = w$.

It follows from the above discussion that *if we restrict the domain of the exponential then it becomes injective.* In fact, if $H = \{z = x + iy : -\pi < y \leq \pi\}$ then $z \to e^z$ is a bijective function from H to $\mathbb{C} \setminus \{0\}$.

We can also understand now image of certain subsets of H under the exponential. For example, for a fixed $y_0 \in (-\pi, \pi]$ if $A = \{x+iy_0 : x \in \mathbb{R}\}$ (which is a line parallel to the real axis) then its image under exponential is $\{e^x e^{iy_0} : x \in \mathbb{R}\}$ which is a one sidded ray with angle y_0. If for a fixed $x_0 \in \mathbb{R}$, $B = \{x_0 + iy : y \in (-\pi, \pi]\}$ (which is part of a line parallel to the imaginary axis) then its image under exponential is $\{e^{x_0} e^{iy} : y \in (-\pi, \pi]\}$ which is a circle about origin with radius e^{x_0}.