LECTURE 5: COMPLEX LOGARITHM AND TRIGONOMETRIC FUNCTIONS

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Recall that $\exp : \mathbb{C} \to \mathbb{C}^*$ is surjective (onto), that is, given $w \in \mathbb{C}^*$ with $w = \rho(\cos\phi + i\sin\phi)$, $\rho = |w|$, $\phi = \text{Arg } w$ we have $e^z = w$ where $z = \ln\rho + i\phi$ (ln stands for the real log) Since exponential is not injective (one one) it does not make sense to talk about the inverse of this function. However, we also know that $\exp : H \to \mathbb{C}^*$ is bijective. So, what is the inverse of this function? Well, that is the logarithm. We start with a general definition

Definition 1. *For* $z \in \mathbb{C}^*$ *we define* $\log z = \ln|z| + i$ *argz*.

Here $\ln|z|$ stands for the real logarithm of $|z|$. Since $\arg z = \text{Arg} z + 2k\pi$, $k \in \mathbb{Z}$ it follows that $\log z$ is not well defined as a function (*it is multivalued*), which is something we find difficult to handle. It is time for another definition.

Definition 2. *For* $z \in \mathbb{C}^*$ *the principal value of the logarithm is defined as Log* $z = \ln|z| + i$ *Argz*.

Thus the connection between the two definitions is $\text{Log } z + 2k\pi = \log z$ for some $k \in \mathbb{Z}$. Also note that $\text{Log} : \mathbb{C}^* \to H$ is well defined (*now it is single valued*).

Remark: We have the following observations to make,

(1) If $z \neq 0$ then $e^{\text{Log } z} = e^{\ln|z| + i \text{ Arg} z} = z$ (What about Log (e^z)?).

(2) Suppose x is a positive real number then $\text{Log } x = \ln x + i \text{ Arg} x = \ln x$ (*for positive real numbers we do not get anything new*).

(3) $\text{Log } i = \ln|i| + i\frac{\pi}{2} = \frac{i\pi}{2}$, $\text{Log } (-1) = \ln|-1| + i\pi = i\pi$,
$\text{Log } (-i) = \ln|-i| + i\frac{-\pi}{2} = -\frac{i\pi}{2}$, $\text{Log } (-e) = 1 + i\pi$ (check!))

(4) The function Log z is not continuous on the negative real axis $\mathbb{R}^- = \{z = x + iy : x < 0, y = 0\}$ (*Unlike real logarithm, it is defined there, but useless*). To see this consider the point $z = -\alpha$, $\alpha > 0$. Consider the sequences $\{a_n = \alpha e^{i(\pi - \frac{1}{n})}\}$ and $\{b_n = \alpha e^{i(-\pi + \frac{1}{n})}\}$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = z$ but $\lim_{n\to\infty} \text{Log } a_n = \lim_{n\to\infty} \ln\alpha + i(\pi - \frac{1}{n}) = \ln\alpha + i\pi$ and $\lim_{n\to\infty} \text{Log } b_n = \ln\alpha - i\pi$.

(5) $z \to \text{Log } z$ is analytic on the set $\mathbb{C}^* \setminus \mathbb{R}^-$. Let $z = re^{i\theta} \neq 0$ and $\theta \in (-\pi, \pi)$. Then $\text{Log } z = \ln r + i\theta = u(r,\theta) + iv(r,\theta)$ with $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$.

Then $u_r = \frac{1}{r}v_\theta = \frac{1}{r}$ and $v_r = -\frac{1}{r}u_\theta$. Thus the CR equations are satisfied. Since $u_r, u_\theta, v_r, v_\theta$ are continuous the result follows from a previous theorem regarding converse of CR equations.

(6) The identity $\text{Log}\,(z_1z_2) = \text{Log} z_1 + \text{Log}\, z_2$ is not always valid. However, the above identity is true iff $\text{Arg}\, z_1 + \text{Arg}\, z_2 \in (-\pi, \pi]$ (why?).

In calculus, interesting examples of differentiable functions, apart from polynomials and exponential, are given by trigonometric functions. The situation is similar for functions of complex variables.

If $x \in \mathbb{R}$ then using Taylor series for sine and cosine we get

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \cos x + i \sin x.$$

Taking clue from the above, we now define

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

It is easy to see by ratio test that the radius of convergence of these two power series is ∞. It now follows easily that $e^{iz} = \cos z + i \sin z$ (Euler's formula) and hence

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Using the above formulae the following theorem follows easily.

Theorem 3. *For any $z \in \mathbb{C}$*

(1) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\sin(z + 2k\pi) = \sin z$, $\cos(z + 2k\pi) = \cos z$, $\sin^2 z + \cos^2 z = 1$.

(2) $\frac{d}{dz}(\sin z) = \cos z$, $\frac{d}{dz}(\cos z) = -\sin z$.

(3) $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$, *and* $\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$ *where* $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$.

There is an important difference between real and complex sine functions. *Unlike the real sine function the complex sine function is unbounded.* To see this notice that $|\sin z|^2 = |\sin(x + iy)|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x + \sinh^2 y$. As $\lim_{y\to\infty} \sinh y = \infty$ (check this!) it follows that for each fixed $x_0 \in \mathbb{R}$, $\lim_{y\to\infty} |\sin(x_0 + iy)| = \infty$. Similar is the case for $\cos z$.

Now using sine and cosine we can define $\tan z$, $\sec z$, $\text{cosec}\, z$ as in the real case. We can also define complex analogue of the hyperbolic functions $\sinh z = (e^z - e^{-z})/2$

and $\cosh z = (e^z + e^{-z})/2$. The following theorem follows just by applying the definitions

Theorem 4. (1) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$, $\sin 2z = 2 \sin z \cos z$, $\sin(z + \pi) = -\sin z$, $\sin(z + 2\pi) = \sin z$.

(2) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$, $\cos 2z = \cos^2 z - \sin^2 z$.