LECTURE 6: COMPLEX INTEGRATION

The point of looking at complex integration is to understand more about analytic functions. In the process we will see that *any analytic function is infinitely differentiable* and *analytic functions can always be represented as a power series.* We will also be able to give you a proof of the well known *fundamental theorem of algebra*

Integral of a complex valued function of real variable:

Let $f : [a, b] \to \mathbb{C}$ be a function. Then $f(t) = u(t) + iv(t)$ where $u, v : [a, b] \to \mathbb{C}$. We then define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

If $U' = u$ and $V' = v$ and $F(t) = U(t) + iV(t)$ the it follows from the fundamental theorem of calculus that $\int_a^b f(t)dt = F(b) - F(a)$.

Example 1. (1) *For* $\alpha \in \mathbb{R}$, $\int_a^b e^{i\alpha t}dt = \frac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}$.

(2) *Evaluate* $\int_0^{\frac{\pi}{2}} e^t \cos t dt$.

Instead of thinking about integration by parts we use the fact $e^{it} = \cos t + i \sin t$. *Consider the integral* $\int_0^{\frac{\pi}{2}} e^{t+it}dt$. *Thus for* $F(t) = \frac{e^{t(1+i)}}{1+i}$ *we have* $F'(t) = e^{t(1+i)}$ *and hence* $\int_0^{\frac{\pi}{2}} e^{t+it}dt = \frac{ie^{\frac{\pi}{2}} - 1}{1+i} = \frac{e^{\frac{\pi}{2}} - 1}{2} + \frac{i}{2}(e^{\frac{\pi}{2}} + 1)$. *So* $\int_0^{\frac{\pi}{2}} e^t \cos t dt = Re\ (\int_0^{\frac{\pi}{2}} e^{t+it}dt) = \frac{e^{\frac{\pi}{2}} - 1}{2}$.

Definition 2. (1) *A curve* C *in the complex plane* $\mathbb{C}$ *is given by a function* $\gamma : [a, b] \to \mathbb{C}$, $\gamma(t) = x(t) + iy(t)$ *with* $x, y : [a, b] \to \mathbb{R}$ *being continuous. The curve* C *is then the set* $C = \{\gamma(t) : t \in [a, b]\}$.

A curve c *is called a smooth curve if* $\gamma'(t) = x'(t) + iy'(t)$ *is continuous and nonzero for all* t.

A contour is a smooth curve that is obtained by joining finitely many smooth curves end to end.

(2) *(Contour Integral) Let* $C = \gamma(t)$, $t \in [a, b]$ *be a contour and* $f : \mathbb{C} \to \mathbb{C}$ *be continuous then*

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

One can show that the contour integral is independent of the parametrization of the curve C.

Example 3. *(The fundamental integral)*
For $a \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{Z}$

$$\int_{C_{a,r}} (z-a)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

where $C_{a,r}$ denotes the circle of radius r centered at a.

Let $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi)$. Then

$$\int_{C_{a,r}} (z-a)^n dz = \int_0^{2\pi} (re^{it})^n ire^{it} dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt = \frac{ir^{n+1}}{i(n+1)} (e^{i(n+1)2\pi} - 1),$$

which is zero if $n \neq -1$ and if $n = -1$ then from the second integral in the above the result is $2\pi i$

We record the fact, which will be generalized soon: for $n \geq 0$ the integrand is an analytic function and its integral over $C_{a,r}$ is zero.

Next aim is to get an analogue of the fundamental theorem of calculus for contour integrals. We need a definition

Definition 4. *An open subset $D \subseteq \mathbb{C}$ is called connected if any two points of D can be joined by a curve. $D \subseteq \mathbb{C}$ is called a domain if it is open and connected.*

Theorem 5. *Let f be a continuous function defined on a domain D and there exist a function F defined on D such that $F' = f$. Let $z_1, z_2 \in D$. Then for any contour C lying in D starting from z_1, and ending at z_2 the value of the integral $\int_C f(z)dz$ is independent of the contour.*

Proof. Suppose that that C is given by a map $\gamma : [a, b] \to \mathbb{C}$. Then $\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t)$. Hence $\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt = F(\gamma(a)) - F(\gamma(b)) = F(z_2) - F(z_1)$. $\square$

(Note that if $z_1 = z_2$ then the above integral is zero)
Thus in practise, when F exist we can write the expression $\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz$.

Example 6.
(1) $\int_{z_1}^{z_2} z^2 dz = \frac{z_1^3 - z_2^3}{3}$.
(2) $\int_{-i\pi}^{i\pi} \cos z dz = \sin(i\pi) - \sin(-i\pi) = 2\sin(i\pi)$.
(3) $\int_{-i}^{i} \frac{1}{z} dz = Log\ (i) - Log\ (-i) = \frac{i\pi}{2} - \frac{-i\pi}{2} = i\pi$.
(4) *The function $\frac{1}{z^n}$, $n > 1$ is continuous on $\mathbb{C}^*$. Thus the integral of the above function on any contour joining nonzero complex numbers z_1, z_2 not passing through origin is given by $\int_{z_1}^{z_2} \frac{dz}{z^n} = -(n-1)(\frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}})$. In particular we*

have $\int_C \frac{dz}{z^n} = 0$ *where* C *is given by a circle of radius* r *around* 0 *(which we already know from the fundamental integral).*

We need some more (easy!) theorems. Let $\gamma : [a,b] \to \mathbb{C}$ be a curve then the curve with the reverse orientation is denoted as γ_- and is defined as $\gamma_- : [a,b] \to \mathbb{C}$, $\gamma_-(t) = \gamma(b+a-t)$. Thus for a contour C the contour with the negative orientation C_- make sense.

Theorem 7. (1) $\int_{C_-} f(z)dz = -\int_C f(z)dz$.

(2) *Let* $\gamma : [a,b] \to \mathbb{C}$ *be a curve and* $a < c < b$. *If* $\gamma_1 = \gamma|[a,c]$ *and* $\gamma_2 = \gamma|[c,b]$ *then* $\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$.

(3) *Let* f *be a continuous function and gamma be a curve defining a contour* C. *If* $|f(z)| \leq M$ *for all* $z \in C$ *and* l *=length of* γ *then* $|\int_C f(z)dz| \leq Ml$.

Proof. The first two results follow straightway from the definitions involved.

For the last one we can deduce from the definition that

$$|\int_C f(z)dz| \leq \int_a^b |f(\gamma(t)||\gamma'(t)|dt \leq M\int_a^b |\gamma'(t)|dt = Ml.$$

□