# LECTURE 8: CAUCHY'S INTEGRAL FORMULA I

We start by observing one important consequence of Cauchy's theorem: Let $D$ be a simply connected domain and $C$ be a simple closed curve lying in $D$. For some $r > 0$, let $C_r$ be a circle of radius $r$ around a point $z_0 \in D$ lying in the region enclosed by $C$. If $f$ is analytic on $D \setminus \{z_0\}$ then $\int_C f(z)dz = \int_{C_r} f(z)dz$. The proof

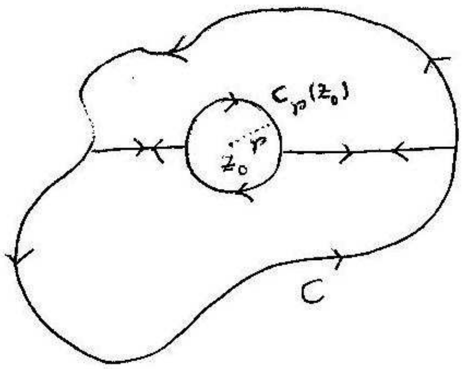


Figure 1

follows by breaking the region into two simply connected domains and by applying Cauchy's theorem (see the figure above).

We have already seen the fundamental integral $\int_{C(0,1)} \frac{1}{z-z_0}dz = 2\pi i$ where $C(0,1)$ is a circle around zero with radius one. We will see that under certain conditions on a function $f$ and a closed curve $C$ one has $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$ which is a generalization of the fundamental integral.

**Theorem 1.** *(Cauchy's integral formula)*

*Let $f$ be analytic on a simply connected domain $D$. Suppose that $z_0 \in D$ and $C$ is a simple closed curve in $D$ that encloses $z_0$. Then*

$$\int_C \frac{f(z)}{z - z_0}dz = 2\pi i f(z_0)$$

*(the above integral is oriented in the counterclockwise sense).*

*Proof.* It follows from a consequence of Cauchy's theorem (see above) that if $C(z_0, r)$ denotes the circle of radius $r$ around $z_0$ for a sufficiently small $r > 0$ then

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz - f(z_0)\right| &= \left|\frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \\
&= \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})-f(z_0)}{re^{i\theta}} ire^{i\theta}d\theta\right| \\
&\leq \frac{1}{2\pi}2\pi \times \sup_{\theta\in[0,2\pi)}|f(z_0+re^{i\theta})-f(z_0)| \\
&\quad (\text{ by } ML \text{ inequality}).
\end{aligned}$$

As $f$ is continuous it follows that the righthand side goes to zero as $r$ tends to zero. This completes the proof.

□

**Example 2.** (1) $\int_{C(4,5)} \frac{\cos z}{z} dz = 2\pi i \cos(0) = 2\pi i$ *(note that the integrand is not analytic in the region enclosed by the curve).*

(2) $\int_{C(i,1)} \frac{z^2}{z^2+1}dz = \int_{C(i,1)} \frac{z^2/(z+i)}{z-i}dz = 2\pi i \frac{i^2}{i+i} = -\pi.$

(3) *The integral* $I = \int_{C(0,2)} \frac{e^z}{z(z-1)}dz$ *cannot be evaluated directly from Cauchy's integral formula but we can rewrite* $I = \int_{C(0,2)} \frac{e^z}{z-1}dz - \int_{C(0,2)} \frac{e^z}{z}dz$ *and apply Cauchy's integral formula to get the value of the integral as* $2\pi i(e-1)$.

Using partial fraction, as we did in the last example, can be a laborious method. We will have more powerful methods to handle integrals of the above kind.

Fortunately Cauchy's integral formula is not just about a method of evaluating integrals. It has more serious theoretical impact. Next one is a very surprising result. Let $f^n$ denotes the $n$-th derivative of $f$.

**Theorem 3.** *If $f$ is analytic on a simply connected domain $D$ then $f$ has derivatives of all orders in $D$ (which are then analytic in $D$) and for any $z_0 \in D$ one has*

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz,$$

*where $C$ is a simple closed contour (oriented counterclockwise) around $z_0$ in $D$.*

*Proof.* (*) Using Cauchy's integral formula we can write that

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0}\frac{1}{2\pi i h}\int_C \left(\frac{f(z)}{z-z_0-h} - \frac{f(z)}{z-z_0}\right)dz \\
&\quad (C \text{ is so chosen that the point } z_0+h \text{ is enclosed by } C) \\
&= \lim_{h\to 0}\frac{1}{2\pi i h}\int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)}dz.
\end{aligned}$$

So we need to prove that

$$\begin{aligned} & \Big| \int_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz - \int_C \frac{f(z)}{(z-z_0)^2}dz \Big| \\ = & \Big| \int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2}dz \Big| \to 0, \text{ as } \quad h \to 0. \end{aligned}$$

We will basically use ML inequality to prove this. Note that, as $f$ is continuous it is bounded on $C$ by $M$ (say). Let $\alpha = \min\{|z-z_0| : z \in C\}$. Then $|z-z_0|^2 \geq \alpha^2$ and $\alpha \leq |z-z_0| = |z-z_0-h+h| \leq |z-z_0-h| + |h|$ and hence for $|h| \leq \frac{\alpha}{2}$ (after all $h$ is going to be small) we get $|z-z_0-h| \geq \alpha - |h| \geq \frac{\alpha}{2}$. Therefore

$$\Big| \int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2}dz \Big| \leq \frac{M|h|l}{\frac{\alpha}{2}\alpha^2} = \frac{2M|h|l}{\alpha^3} \to 0,$$

as $h \to 0$. By repeating exactly the same technique we get $f^2(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz$ and so on.

$\square$

Thus we got an important result which is very different from real analysis: *if $f$ is analytic at a point then all possible derivatives of $f$ are analytic at that point.*

**Example 4.** *Cauchy's integral formula is very convenient for evaluation of some integrals.*

(1) $\int_{\{z:|z|=1\}} e^z z^{-3} dz = \frac{2\pi i}{2}\frac{d^2}{dz^2}(e^z)|_{z=0} = i\pi$ *(notice the power in the denominator and differentiate one time less).*

(2) *If $C$ is a circle of radius $5/2$ around the point $1$ then consider the integral $\int_C \frac{1}{(z-4)(z+1)^4}dz$. As $-1$ is enclosed by $C$ the function $\frac{1}{(z+1)^4}$ is not analytic in the region enclosed by $C$, thus we consider $f(z) = \frac{1}{z-4}$ and apply Cauchy's integral formula to get*

$$\int_C \frac{1}{(z-4)(z+1)^4}dz = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\Big(\frac{1}{z-4}\Big)\Big|_{z=-1}.$$