LECTURE 9: CAUCHY'S INTEGRAL FORMULA II

Let us first summarize Cauchy's theorem and Cauchy's integral formula. Let C be a simple closed curve contained in a simply connected domain D and f is an analytic function defined on D. Then

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \begin{cases} 2\pi i f(z_0), & \text{if } n=0 \text{ and } z_0 \text{ is enclosed by } C. \\ \frac{2\pi i}{n!} f^n(z_0), & \text{if } n \geq 1 \text{ and } z_0 \text{ is enclosed by } C. \\ 0, & z_0 \text{ lies out side the region enclosed by } C. \end{cases}$$

By Cauchy's integral formula one can also tackle integrals of the form $\int_C \frac{f(z)}{(z-z_0)(z-z_1)}dz$ where the simple closed curve C includes two points z_0, z_1. By using partial fraction we get that

$$\begin{aligned} \int_C \frac{f(z)}{(z-z_0)(z-z_1)}dz &= \int_C \frac{f(z)}{z_0-z_1}(\frac{1}{z-z_0} - \frac{1}{z-z_1})dz \\ &= \frac{2\pi i(f(z_0)-f(z_1))}{(z_0-z_1)}. \end{aligned}$$

Example 1. *If $a \in \mathbb{C}$ then*

$$\int_{\{z:|z|=2\}} \frac{e^{az}}{z^2+1}dz = \int_{\{z:|z|=2\}} \frac{e^{az}}{(z+i)(z-i)}dz = \frac{e^{-ia}-e^{ia}}{4\pi}.$$

We will now see some more serious application of CIF. For $r > 0$ let us define $\overline{B_r(z_0)} = \{z : |z-z_0| \leq r\}$ and $S_r(z_0) = \{z : |z-z_0| = r\}$.

Theorem 2. *(Cauchy's estimate) Suppose that f is analytic on a simply connected domain D and $\overline{B_R(z_0)} \subset D$ for some $R > 0$. If $|f(z)| \leq M$ for all $z \in S_R(z_0)$, then for all $n \geq 0$,*

$$|f^n(z_0)| \leq \frac{n!M}{R^n}.$$

Proof. From Cauchy's integral formula and ML inequality we have

$$|f^n(z_0) = |\frac{n!}{2\pi i}\int_{S_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}dz| \leq \frac{n!}{2\pi}M\frac{1}{R^{n+1}}2\pi R = \frac{n!M}{R^n}$$

□

As a consequence of the above theorem we get the following miraculous result.

Theorem 3. *(Liouville's Theorem) If f is analytic and bounded on the whole $\mathbb{C}$ then f is a constant function.*

Proof. To prove this we will prove that f' is the zero function. Choose $\epsilon > 0$ arbitrary and choose any point $z_0 \in \mathbb{C}$. Now consider $\overline{B_R(z_0)}$ such that $R > M/\epsilon$ (for small ϵ, R will be very large but that is not a problem as f is analytic everywhere).By Cauchy's estimate now we have,

$$|f'(z_0)| \leq \frac{M}{R} < \epsilon.$$

Hence $f'(z_0) = 0$. But z_0 is arbitrary and hence $f'(z) = 0$ for all $z \in \mathbb{C}$. □

Remark: We have earlier observed that $\cos z$ and $\sin z$ are not bounded in $\mathbb{C}$. Another proof of the same fact now follows from Liouville's theorem. Moreover it shows that this behavior is typical of non constant analytic functions on $\mathbb{C}$. Thus *if a function is bounded it cannot be analytic on whole* $\mathbb{C}$.

We now show another application of Liouville's theorem to prove the *Fundamental Theorem of Algebra.*

Theorem 4. *Every polynomial $p(z)$ of degree $n \geq 1$ has a root (in $\mathbb{C}$).*

Proof. Suppose $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ is a polynomial with no root in $\mathbb{C}$. Then $\frac{1}{P(z)}$ is analytic on whole $\mathbb{C}$. Since

$$|\frac{P(z)}{z^n}| = |1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}| \to 1, \text{ as } |z| \to \infty,$$

it follows that $|p(z)| \to \infty$ and hence $|1/p(z)| \to 0$ as $|z| \to \infty$ (we are just proving a well known fact that polynomials are unbounded functions). Consequently $\frac{1}{p(z)}$ is a bounded function. Hence by Liouville's theorem $\frac{1}{p(z)}$ is constant which is impossible. □

We will now prove a partial converse to Cauchy's theorem

Theorem 5. *(Morera's theorem) If f is continuous in a simply connected domain D and if $\int_C f(z)dz = 0$ for every simple closed contour C in D then f is analytic*

Proof. The idea is just to prove that there exists an analytic function F such that $F' = f$. Then we can use CIF to conclude that f is analytic. So, fix a point $z_0 \in D$ and define $F(z) = \int_{z_0}^{z} f(w)dw$ (by hypothesis it does not matter which closed curve I use). By using continuity, we can show as before that F is analytic and $F' = f$. □

The next theorem shows that an analytic function is always given by a power series.

Theorem 6. *(Taylor's Theorem)*
Let f be analytic on $D = \{z : |z - z_0| < R_0\}$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{for all } z \in D,$$

where $a_n = \frac{f^n(z_0)}{n!}$ for $n = 0, 1, 2, \ldots$.

Proof. (*) Without loss of generality we consider $z_0 = 0$. Fix $z \in D$. Let $|z| = r$ and C_0 be a circle with center 0 and radius r_0 such that $r < r_0 < R_0$. We need the following algebraic identity,

$$\frac{1}{1-q} = 1 + q + q^2 + \ldots\ldots + q^{(n-1)} + \frac{q^n}{1-q},$$

which follows easily from

$$1 + q + q^2 + \ldots.. + q^{n-1} = \frac{1-q^n}{1-q}.$$

Thus for two complex numbers w and z we can write

$$\frac{1}{w-z} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \ldots. + \frac{z^{n-1}}{w^n} + \frac{z^n}{(w-z)w^n}. \tag{0.1}$$

By CIF and (0.1) we now have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_0} \frac{f(z)dw}{w-z} \\ &= \frac{1}{2\pi i}\int_{C_0} f(w)\left[\frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \ldots. + \frac{z^{n-1}}{w^n}\right]dw + \frac{z^n}{2\pi i}\int_{C_0}\frac{f(w)dw}{(w-z)w^n} \\ &= f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \ldots.. + \frac{f^{n-1}(0)}{(n-1)!}z^{n-1} + \rho_n(z) \end{aligned}$$

where $\rho_n(z) = \frac{z^n}{2\pi i}\int_{C_0}\frac{f(w)dw}{(w-z)w^n}$. Now, we just need to show that $\lim_{n\to\infty}|\rho_n(z)| = 0$. Notice that the function $w \to \frac{f(w)}{w-z}$ is a bounded function on the circle C_0 (as it is continuous). Thus by ML inequality it follows that

$$|\rho_n(z)| \le K r_0 \left|\frac{z}{r_0}\right|^n.$$

As $|z| = r < r_0$ it follows that the right hand side goes to zero as $n \to \infty$. ☐