MTH 111-2023

Assignment 1 : Real Numbers, Sequences

1. Find the supremum of the set $\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\}$.

2. Let A be a non-empty subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$. Show that $\alpha = supA$ if and only if $\alpha - \frac{1}{n}$ is not an upper bound of A but $\alpha + \frac{1}{n}$ is an upper bound of A for every $n \in \mathbb{N}$.

3. Let $y \in (1, \infty)$ and $x \in (0, 1)$. Evaluate $\lim_{n\to\infty}(2n)^y x^n$.

4. For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \frac{1}{4}(x_n^2 + 3)$ for all $n \in \mathbb{N}$. Show that (x_n) converges if and only if $|a| \leq 3$. Moreover, find the limit of the sequence when it converges.

5. Show that the sequence (x_n) defined by $x_1 = \frac{1}{2}$ and $x_{n+1} = \frac{1}{7}\left(x_n^3 + 2\right)$ for $n \in \mathbb{N}$ satisfies the Cauchy criterion.

6. Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ for $n \in \mathbb{N}$. Show that $|x_{2n} - x_n| \geq \frac{1}{2}$ for every $n \in \mathbb{N}$. Does the sequence (x_n) satisfy the Cauchy criterion ?

7. Let (x_n) be defined by $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{x_n + x_{n+1}}{2}$ for $n \geq 1$. Show that (x_n) converges. Further, by observing that $x_{n+2} + \frac{x_{n+1}}{2} = x_{n+1} + \frac{x_n}{2}$, find the limit of (x_n).

Assignment 2 : Continuity, Existence of minimum, Intermediate Value Property

1. Let $[x]$ denote the integer part of the real number x. Suppose $f(x) = g(x)h(x)$ where $g(x) = [x^2]$ and $h(x) = \sin 2\pi x$. Discuss the continuity/discontinuity of f, g and h at $x = 2$ and $x = \sqrt{2}$.

2. Determine the points of continuity for the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational.} \end{cases}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $x_0, c \in \mathbb{R}$. Show that if $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

4. Let $f : [0, 1] \to (0, 1)$ be an on-to function. Show that f is not continuous on $[0, 1]$.

5. Let $f : [a, b] \to \mathbb{R}$ and for every $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| < \frac{1}{2}|f(x)|$. Find $\inf\{|f(x)| : x \in [a, b]\}$. Show that f is not continuous on $[a, b]$.

6. Let $f : [0, 2] \to \mathbb{R}$ be a continuous function and $f(0) = f(2)$. Prove that there exist real numbers $x_1, x_2 \in [0, 2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

7. Let p be an odd degree polynomial and $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$. Further show that the equation $x^{13} - 3x^{10} + 4x + sinx = \frac{1}{1+x^2} + cos^2 x$ has a solution in $\mathbb{R}$.

Assignment 3 : Derivatives, Maxima and Minima, Rolle's Theorem

1. Show that the function $f(x) = x \mid x \mid$ is differentiable at 0. More generally, if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0.

2. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an even function (i.e., $f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative f' is an odd function (i.e.,$f(-x) = -f(x)$ for all $x \in \mathbb{R}$).

3. Show that among all triangles with given base and the corresponding vertex angle, the isosceles triangle has the maximum area.

4. Show that exactly two real values of x satisfy the equation $x^2 = xsinx + cosx$.

5. Suppose f is continuous on $[a,\ b]$, differentiable on $(a,\ b)$ and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. Then show that the equation $f'(x)f(x) = x$ has at least one root in $(a,\ b)$.

6. Let $f : (-1, 1) \to \mathbb{R}$ be twice differentiable. Suppose $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Show that $f'(0) = f''(0) = 0$.

7. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that $f''(0) > 0$. Show that there exists $n \in \mathbb{N}$ such that $f(\frac{1}{n}) \neq 1$.

Assignment 4 : Mean Value Theorem, Taylor's Theorem, Curve Sketching

1. Show that $ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$ if $0 < y \leq x$, $n \in \mathbb{N}$.

2. Let $f : [0, 1] \to \mathbb{R}$ be differentiable, $f(\frac{1}{2}) = \frac{1}{2}$ and $0 < \alpha < 1$. Suppose $|f'(x)| \leq \alpha$ for all $x \in [0, 1]$. Show that $|f(x)| < 1$ for all $x \in [0, 1]$.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$ and $f(b) = b$. Show that there is $c \in (a, b)$ such that $f'(c) = 1$. Further, show that there are distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

4. Using Cauchy Mean Value Theorem, show that
   (a) $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.
   (b) $x - \frac{x^3}{3!} < \sin x$ for $x > 0$.

5. Find $\lim\limits_{x \longrightarrow 5} (6 - x)^{\frac{1}{x-5}}$ and $\lim\limits_{x \longrightarrow 0^+} (1 + \frac{1}{x})^x$.

6. Sketch the graphs of $f(x) = x^3 - 6x^2 + 9x + 1$ and $f(x) = \frac{x^2}{x^2-1}$.

7. (a) Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x_0 \in [a, b]$. Show that for any $x \in [a, b]$
   $$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$
   i.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.
   (b) Show that $\cos y - \cos x \geq (x - y) \sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

8. Suppose f is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem show that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.

Assignment 5 : Series, Power Series, Taylor Series

1. Let $f : [0, 1] \to \mathbb{R}$ and $a_n = f(\frac{1}{n}) - f(\frac{1}{n+1})$. Show that if f is continuous then $\sum_{n=1}^{\infty} a_n$ converges and if f is differentiable and $|f'(x)| < 1$ for all $x \in [0, 1]$ then $\sum_{n=1}^{\infty} |a_n|$ converges.

2. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$ where a_n equals:

(a) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$ (b) $1-\cos\frac{1}{n}$ (c) $2^{-n-(-1)^n}$ (d) $\left(1+\frac{1}{n}\right)^{n(n+1)}$

(e) $\frac{n\ln n}{2^n}$ (f) $\frac{logn}{n^p}, (p>1)$ (g) $e^{-n}(\cos n)n^2\sin\frac{1}{n}$

3. Let $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n}\le\frac{b_{n+1}}{b_n}$ for all $n\ge N$. Show that if $\sum\limits_{n=1}^{\infty} b_n$ converges then $\sum\limits_{n=1}^{\infty} a_n$ also converges. Test the series $\sum\limits_{n=1}^{\infty}\frac{n^{n-2}}{e^n n!}$ for convergence.

4. Show that the series $\frac{1}{4^1}+\frac{1}{5^2}+\frac{3}{4^3}+\frac{1}{5^4}+\frac{5}{4^5}+\frac{1}{5^6}+\frac{7}{4^7}+\cdots$ converges.

5. Show that the series $\sum_{n=1}^{\infty}(-1)^n\sin\frac{1}{n}$ converges but not absolutely.

6. Determine the values of x for which the series $\sum\limits_{n=1}^{\infty}\frac{(x-1)^{2n}}{n^2 3^n}$ converges.

7. Show that $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},\ x\in\mathbb{R}$.

Assignment 6: Integration

1. Using Riemann's criterion for the integrability, show that $f(x)=\frac{1}{x}$ is integrable on $[1,2]$.

2. If f and g are continuous functions on $[a,\ b]$ and if $g(x)\ge 0$ for $a\le x\le b$, then show the mean value theorem for integrals : there exists $c\in[a,\ b]$ such that $\int\limits_a^b f(x)g(x)dx=f(c)\int\limits_a^b g(x)dx$.

   (a) Show that there is no continuous function f on $[0,1]$ such that $\int\limits_0^1 x^n f(x)dx=\frac{1}{\sqrt{n}}$ for all $n\in\mathbb{N}$.

   (b) If f is contiunuous on $[a,b]$ then show that there exists $c\in[a,b]$ such that $\int_a^b f(x)dx=f(c)(b-a)$.

   (c) If f and g are continuous on $[a,b]$ and $\int_a^b f(x)dx=\int_a^b g(x)dx$ then show that there exists $c\in[a,b]$ such that $f(c)=g(c)$.

3. Let $f:[0,2]\to\mathbb{R}$ be a continuous function such that $\int_0^2 f(x)dx=2$. Find the value of $\int_0^2[xf(x)+\int_0^x f(t)dt]dx$.

4. Show that $\int\limits_0^x(\int\limits_0^u f(t)dt)du=\int\limits_0^x f(u)(x-u)du$, assuming f to be continuous.

5. Let $f:[0,1]\to\mathbb{R}$ be a positive continuous function. Show that $\lim_{n\to\infty}(f(\frac{1}{n})f(\frac{2}{n})\cdots f(\frac{n}{n}))^{\frac{1}{n}}=e^{\int_0^1 lnf(x)}$.

Assignment 7: Improper Integrals

1. Test the convergence/divergence of the following improper integrals:

   (a) $\int\limits_0^1\frac{dx}{\log(1+\sqrt{x})}$ (b) $\int\limits_0^1\frac{dx}{x-\log(1+x)}$ (c) $\int\limits_0^1\frac{\log x}{\sqrt{x}}$ (d) $\int\limits_0^1\sin(1/x)dx.$

   (e) $\int\limits_1^{\infty}\frac{\sin(1/x)}{x}dx$ (f) $\int\limits_0^{\infty}e^{-x^2}dx$ (g) $\int\limits_0^{\infty}\sin x^2dx,$ (h) $\int\limits_0^{\pi/2}\cot xdx.$

2. Determine all those values of p for which the improper integral $\int_0^{\infty}\frac{1-e^{-x}}{x^p}dx$ converges.

3. Show that the integrals $\int\limits_0^\infty \frac{\sin^2 x}{x^2} dx$ and $\int\limits_0^\infty \frac{\sin x}{x} dx$ converge. Further, prove that $\int\limits_0^\infty \frac{\sin^2 x}{x^2} dx = \int\limits_0^\infty \frac{\sin x}{x} dx$.

4. Show that $\int\limits_0^\infty \frac{x \log x}{(1+x^2)^2} dx = 0$.

5. Prove the following statements.

   (a) Let f be an increasing function on (0,1) and the improper integral $\int_0^1 f(x)$ exist. Then

      i. $\int_0^{1-\frac{1}{n}} f(x)dx \leq \frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-1}{n})}{n} \leq \int_{\frac{1}{n}}^1 f(x)dx$.

      ii. $\lim_{n\to\infty} \frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-1}{n})}{n} = \int_0^1 f(x)dx$.

   (b) $\lim_{n\to\infty} \frac{ln\frac{1}{n}+ln\frac{2}{n}+\cdots+ln\frac{n-1}{n}}{n} = -1$.

   (c) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$.