Lecture 1: The Real Number System

In calculus of a single variable, we study continuity, differentiability and integration of functions whose domain is either the set of all real numbers or its subset. In order to study these concepts, we need some properties of real number system. These properties will be discussed in the first four lectures.

In this lecture, we give some idea about the real number system and present some of its important properties.

We denote the set $\{1, 2, 3, ...\}$ of natural numbers by $\mathbb{N}$ and the set $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ of integers by $\mathbb{Z}$. We know that if p and q are integers, then $p + q$, $p - q$ and the product pq are again integers. Further, if $p - q > 0$, then we say that $p > q$ and this defines an order relation '$>$' on $\mathbb{Z}$.

Next we consider the numbers of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$, called rational numbers. The set of rational numbers is denoted by $\mathbb{Q}$. Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ and the operations of addition (and subtraction), multiplication and '$<$' defined on $\mathbb{Z}$ can be extended to $\mathbb{Q}$ in a natural way. We can see, from a very simple situation that numbers other than rational numbers are needed. For instance, consider a square whose side has unit length. Then by the Pythagoras Theorem, the length l of the diagonal satisfy $l^2 = 2$ and, in this case, we write $l = \sqrt{2}$. Is l rational? Suppose $l = m/n$, where m and n are integers which are not both even. Then $l^2n^2 = 2n^2 = m^2$. Thus m^2 is even. Since the square of an odd integer is odd, we conclude that m is even and hence n^2 is even. Therefore n is divisible by 2. This contradicts our assumption.

The above discussion shows that $\sqrt{2}$ is not a rational number and we need numbers such as $\sqrt{2}$. Hence $\mathbb{Q}$ needs to be extended, by including new numbers with $\mathbb{Q}$, to form a new number system. What would be the those new numbers? Generally when this is asked in the class, natural reaction from some students would be "the set of irrationals". But, what are those "irrationals" exactly and how to define these from $\mathbb{Q}$? Since the answers to these questions *are bit involved*, we will not present the formal definition and the construction of real numbers here. Instead, we give an overall idea about the real numbers.

Let us look at the geometrical representation of the rational numbers to get an idea. On a straight line, if we mark a point to represent 0 and another, to the right of the marked point, to represent 1, then all the rational numbers can be represented by points on this straight line. It seems that the set of points representing rational numbers fills up this line, because, given any two rational numbers r and s, the rational number $\frac{r+s}{2}$ lies in between. But we have seen above that the rationals leave certain gaps on the line. Hence, intuitively we feel that there should a larger set of numbers, say $\mathbb{R}$, such that every element in the line corresponds to a number in $\mathbb{R}$ and vice versa. Indeed, *there exists* such a set of numbers, called set of real numbers, which contains $\mathbb{Q}$. Moreover, on this set $\mathbb{R}$ we can define the operations of addition and multiplication, and an order in such a way that when these operations and the order are restricted to the set of rationals, they coincide with the usual operations and the usual order. Further, the familiar properties of the addition (and subtraction), multiplication (and division by non-zero numbers) and the order known for $\mathbb{Q}$ can also be extended to $\mathbb{R}$. For instance, one such property is $\alpha(a + b) < \alpha(c + d) = \alpha c + \alpha d$ if a, b, c, d and α are in $\mathbb{R}$ and $a < c, b < d$ and $0 < \alpha$. Apart from these operations and the order defined on $\mathbb{R}$, we need a property on $\mathbb{R}$ which articulates that $\mathbb{R}$ does not leave any gap on the line unlike $\mathbb{Q}$. Such a property is called *completeness property* which is defined below.

For us, the real number system means the set $\mathbb{R}$ with the operations addition and multiplication,

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the order relation and the completeness property. Assuming the completeness property as an axiom, the other required properties of $\mathbb{R}$ will be derived in this and the subsequent lectures.

Completeness or least upper bound property: *Every non-empty subset of* $\mathbb{R}$ *which is bounded above has a least upper bound.*

What does this mean? This will be clear once if we define the terms "bounded above", "upper bound" and "least upper bound".

Definition 1.1. A subset A of $\mathbb{R}$ is said to be *bounded above* if there is an element $\alpha \in \mathbb{R}$ such that $x \leq \alpha$ for all $x \in A$. Every such an element α is called an *upper bound* of A.

Example 1.1. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. This set is bounded above. For instance, 3, $\frac{5}{4}$, 2 and 1 are upper bounds. In fact, any $\alpha \in \mathbb{R}$ such that $\alpha \geq 1$ is an upper bound.

In the proceeding example, 1 is not only an upper bound of A and it seems that it is the "least" among all the upper bounds of A. Let us define the term "least" formally.

Definition 1.2. Let A be a subset of $\mathbb{R}$. An element $\beta \in \mathbb{R}$ is said to be a *least upper bound (l.u.b.)* or *supremum* of A if it satisfies the folowing two conditions:

(i) β is an upper bound of A;

(ii) if α is any upper bound for A, then $\beta \leq \alpha$.

We have seen in Example 1.1 that a set can have several upper bounds. But a set can have only one least upper bound. Suppose that β_1 and β_2 are two least upper bounds of A. Then by condition (ii) of Definition 1.2, $\beta_1 \leq \beta_2$ and $\beta_2 \leq \beta_1$ which concludes that $\beta_1 = \beta_2$.

Example 1.2. 1. Let A be defined as in Example 1.1. We show that 1 is the least upper bound of A. We take $\beta = 1$ and verify that β satisfies conditions (i) and (ii) of Definition 1.2. It is clear that 1 is an upper bound of A. To verify (ii), let α be an upper bound of A. Then $1 \leq \alpha$ because $1 \in A$ and α is an upper bound for A.

2. Let $A = \{x \in \mathbb{R} : x < 1\}$. We claim that the least upper bound of A is 1. Take $\beta = 1$. It is clear that β satisfies condition (i) of Definition 1.2 and let us verify (ii). Suppose β does not satisfy (ii). Then there exists an upper bound α of A such that $\alpha < \beta = 1$. Then $\frac{\alpha+1}{2} \in A$ and $\alpha < \frac{\alpha+1}{2}$. This contradicts the fact that α is an upper bound of A. Hence β is the l.u.b. of A.

The least upper bound of a set, if it exists, may or may not belong to the set (see the sets defined in Examples 1.1 and 1.2).

The notions lower bound, bounded below and greatest lower bound (or infimum) for a subset of $\mathbb{R}$ are defined in a similar way (define!). We denote the least upper bound (supremum) of a set A by $\sup A$ and the infimum by $\inf A$. The least upper bound property implies the greatest lower bound property: *Every non-empty bounded below subset of* $\mathbb{R}$ *has the greatest lower bound* (See Problem 10 in Practice Problems 1 (PP1)).

Remark 1.1.(*) The set of real numbers $\mathbb{R}$ can be constructed from $\mathbb{Q}$ and it can also be shown that $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ has the completeness property.

The next result which will be used later is an important consequence of the completeness property of $\mathbb{R}$. It essentially says that no element of $\mathbb{R}$ can be an upper bound of $\mathbb{N}$. This looks obvious. If so, then why to prove it? Recall that we have not defined $\mathbb{R}$ but we have assumed the existence of such a set with the operations addition and multiplication, the order relation and the completeness property. If we look at Example 1.2, we have used the properties of the said operations and the order to show that the element 1 is the supremum. The next result cannot be derived only from the properties of these operations and the order. The completeness property is also required for the proof.

Theorem 1.1 (Archimedean property). *If $x, y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{N}$, such that $nx > y$.*

Proof (*). Suppose the conclusion fails. Then $nx \le y$ for every $n \in \mathbb{N}$. Let $A = \{nx : n \in \mathbb{N}\}$. Then y is an upper bound of A. Use the l.u.b. property and let α be the l.u.b. of A. Since α is also an upper bound of A and $(n+1)x \in A$, $(n+1)x \le \alpha$ for all n and so $nx \le \alpha - x < \alpha$ for all n. Hence $\alpha - x$ is also an upper bound which is smaller than α. This is not possible, because, α is the l.u.b. of A. □

We now present some consequences of the Archimedean property.

Example 1.3. 1. Let $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ and $\beta = 1$. We claim that β is the l.u.b. of A. As β is an upper bound of A, it is enough to show that β satisfies condition (ii) of Definition 1.2. Suppose β fails to satisfy (ii). Then there exists an upper bound α of A such that $\alpha < \beta = 1$. Using Archimedean Property, we find some $n \in \mathbb{N}$ such that $\alpha < 1 - \frac{1}{n}$, that is, $n(1-\alpha) > 1$. Hence α is not an upper bound of A which is a contradiction.

2. (*) We have already pointed out in the beginning of this lecture that $x^2 = 2$ has no solution in $\mathbb{Q}$. We show that this equation has a solution in $\mathbb{R}$ using the completeness property. Let $A = \{r \in \mathbb{R} : r \ge 0, r^2 < 2\}$. Then A is non-empty as $1 \in A$ and it is bounded above as 2 is an upper bound of A. Use the completeness property and let $\alpha = \sup A$. Using the Archimedean property we can (see Problem 9 of PP1) show that $\alpha^2 = 2$. The same argument shows that if $x_0 > 0$, then there exists a unique $\alpha \in \mathbb{R}$ such that $\alpha^2 = x_0$. In this case we write $\alpha = \sqrt{x_0}$. Suppose $x_0 > 0$. For given $n \in \mathbb{N}$, involving the binomial theorem, we can also show that there exists a unique $\beta \in \mathbb{R}$ such that $\beta^n = x_0$ (a different proof of this fact will be discussed in Lecture 6). In this case we write $x_0^{\frac{1}{n}} = \beta$. If $m \in \mathbb{N}$, then we define $x_0^{\frac{m}{n}} = \beta^m$ and $x_0^{-\frac{m}{n}} = 1/x_0^{\frac{m}{n}}$.

The following two results which will be used later are consequences of the Archimedean property.

Theorem 1.2 (Density of rationals in $\mathbb{R}$). *Between any two distinct real numbers there is a rational number.*

Proof. Suppose $x, y \in \mathbb{R}$ and $y - x > 0$. We have to find two integers m and n, $n \neq 0$ such that $x < \frac{m}{n} < y$, that is, $x < \frac{m}{n} < x + (y - x)$. We first find the denominator n and then the numerator m. First, using the Archimedean property, find $n \in \mathbb{N}$ such that $n(y-x) > 1$. Next, we can find (see Problem 8 of PP1) an integer m lying between nx and $ny = nx + n(y-x)$, because, the difference between these two numbers is greater than 1. This proves the result. □

The numbers in $\mathbb{R}$ which are not rationals are called irrationals. The next result is a consequence of the proceeding result.

Corollary 1.1 (Density of irrationals in $\mathbb{R}$). *Between any two distinct real numbers there is an irrational number.*

Proof. Suppose $x, y \ge 0$ and $x < y$. Then $\sqrt{2} + x < \sqrt{2} + y$. By Theorem 1.2, there exists a rational number r such that $\sqrt{2} + x < r < \sqrt{2} + y$, that is, $x < r - \sqrt{2} < y$. Note that $r - \sqrt{2}$ is an irrational number. □