Lecture 10 : Tests for Local Maximum and Minimum, Point of Inflection, Curve Sketching

In this lecture, we assume $f : (a, b) \to \mathbb{R}$ unless specified otherwise. In Theorem 7.1, we have seen a necessary condition for a point $x_0 \in (a, b)$ to be a local maximum or local minimum of f. The necessary condition is useful for identifying the points which are candidates for points of local maximum or local minimum. We now present some sufficient conditions which can be used to identify the points of local maximum or minimum among those candidates.

Sufficient conditions for local maximum and local minimum

We present sufficient conditions only for local maximum. The corresponding sufficient conditions for local minimum are derived similarly. The following result is anticipated.

Theorem 10.1. *Let $c \in (a, b)$ and f be continuous at c. If for some $\delta > 0$, f is increasing on $(c - \delta, c)$ and decreasing on $(c, c + \delta)$, then c is a point of local maximum of f.*

Proof. Let $x_1 \in (c - \delta, c)$. Using the continuity of f at c, we show that $f(x_1) \leq f(c)$. Choose any x such that $x_1 < x < c$. Then $f(x_1) \leq f(x)$ and by the continuity of f at c we have

$$f(x_1) \leq \lim_{x \to c^-} f(x) = f(c).$$

Similarly, if $x_2 \in (c, c + \delta)$ then $f(x_2) \leq \lim_{x \to c^+} f(x) = f(c)$. This proves the result. $\square$

The following result is considered as the first derivative test for local maximum.

Corollary 10.1. *Let $c \in (a, b)$ and f be continuous at c. If there exists $\delta > 0$ such that*

$$f'(x) \geq 0 \text{ for all } x \in (c - \delta, c) \text{ and } f'(x) \leq 0 \text{ for all } x \in (c, c + \delta)$$

then c is a point of local maximum of f.

Proof. The proof is immediate from Application 7.2 and Theorem 10.1. $\square$

The next result is considered as the second derivative test for local maximum.

Corollary 10.2. *Let $c \in (a, b)$. If $f'(c) = 0$ and $f''(c) < 0$ then c is a point of local maximum of f.*

Proof (*). Since $f''(c)$ exists, $f'(x)$ exists in a neighborhood of c. As $f''(c) < 0$ and $f'(c) = 0$,

$$f''(c) = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x \to c} \frac{f'(x)}{x - c} < 0.$$

Therefore, there exists $\delta > 0$ (see Problem 4 of PP6) such that

$$\frac{f'(x)}{x - c} < 0 \text{ for all } x \in (c - \delta, c) \cup (c, c + \delta).$$

This implies that $f'(x) > 0$ for all $x \in (c - \delta, c)$ and $f'(x) < 0$ for all $x \in (c, c + \delta)$. Apply Corollary 10.1. $\square$

Remark 10.1 We illustrate below that the converses of the preceding results are not true.

(i) If f is continuous at c and c is point of local maximum of f then f need not be increasing on $(c - \delta, c)$ or decreasing on $(c, c + \delta)$ for any $\delta > 0$. Consider $f(x) = -(x \sin(1/x))^2$ if $x \neq 0$,

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$f(0) = 0$ and $c = 0$. If we take any δ-neighborhood around 0, f is neither increasing on $(c - \delta, c)$ nor decreasing on $(c, c + \delta)$. This can be verified as follows. If f is decreasing on $(c, c + \delta)$ for some $\delta > 0$, then there cannot a zero of f on $(c, c + \delta)$. But $f(\frac{1}{n\pi}) = 0$ for every $n \in \mathbb{N}$ which is a contradiction. Hence f is not decreasing on $(c, c + \delta)$. However, $c = 0$ is a point of local maximum of f.

(ii) If c is a point of local maximum of f and f is twice differentiable at c, then $f''(c)$ need not be less than 0. Consider, for example, $f(x) = -x^4$ and $c = 0$.

So, the conditions assumed in the preceding results are sufficient but not necessary.

Example 10.1 1. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^4 e^{-x^2}$. To find the points of local maximum and minimum of f, consider $f'(x) = 2x^3 e^{-x^2}(4 - 2x^2)$ and note that f' vanishes at $x = -\sqrt{2}, \sqrt{2}, 0$ which are the candidates for the points of local maximum and minimum. Note that the sign of f' changes from $+$ to $-$ at $x = -\sqrt{2}$ as well as at $x = \sqrt{2}$. Apply Corollary 10.1 (for f on, for instance, (-2,2)) to conclude that $\sqrt{2}$ and $-\sqrt{2}$ are points of local maximum. Since the sign of f' changes from $-$ to $+$ at $x = 0$, 0 is a point of local minimum. Although the statement of the second derivative test looks simpler, but sometimes, computing the second derivative and using the second derivative test can be complicated (see also Problems 4 and 6 in PP 10).

2. Consider $f : [-2, 5] \to \mathbb{R}$ defined by $f(x) = x^2|x - 3|$. Note that $f(x) = x^2(3 - x)$ on $(-2, 3)$, $f(x) = x^2(x - 3)$ on $(3, 5)$ and f is not differentiable at $x = 3$. Observe that the Corollaries 10.1 and 10.2 are applicable for f defined over $(-2, 3)$ and $(3, 5)$. On $(-2, 3)$, $f'(x) = 3x(2 - x)$ and on $(3, 5)$, $f'(x) = 3x(x - 2)$. Hence the points $-2, 0, 2, 3$ and 5 are the candidates for points of local maximum and minimum. Verify that $0, 3$ are points of local minimum and $-2, 2, 5$ are the points of local maximum for f.

We now introduce the notions convexity, concavity, point of inflection and asymptote. These notions are useful for sketching the graph of a given function.

Convexity, Concavity and Point of Inflection

In this course, we define the convexity and concavity for differentiable functions.

Definition 10.1. Let $f : (a, b) \to \mathbb{R}$ be differentiable. The function f is said to be convex (resp., concave) on (a, b) if f' is strictly increasing (resp., strictly decreasing) on (a, b).

Example 10.2. The function $f(x) = x^2$ on any open interval (in fact on all of $\mathbb{R}$) and the function $f(x) = \sin x$ on $(\pi, 2\pi)$ are convex functions. The function $f(x) = -x^2$ on any open interval and the function $f(x) = \sin x$ on $(0, \pi)$ are concave functions.

It is clear that if f is twice differentiable on (a, b) and $f''(x) > 0$ for all $x \in (a, b)$ then f is convex. A similar result also holds for concavity. The following two observations explain about the shape of the graph of f when $f''(x) > 0$ for all $x \in (a, b)$.

1. It is observed in Application 10.1 that if $f''(x) > 0$ for all $x \in (a, b)$ and $x_0 \in (a, b)$ then the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.

2. It is noted in Application 9.1 that if $f''(x) > 0$ for all $x \in (a, b)$ and $x, y \in (a, b)$ then the chord joining the two points $(x, f(x))$ and $(y, f(y))$ lies above the portion $\{(t, f(t)) : t \in (x, y)\}$ of the graph.

Definition 10.2. Let $f : (a, b) \to \mathbb{R}$ be continuous at a point $c \in (a, b)$. The point c is said to be a point of inflection if there exists a $\delta > 0$ such that either

$$f \text{ is convex on } (c-\delta, c) \text{ and } f \text{ is concave on } (c, c+\delta)$$

or

$$f \text{ is concave on } (c-\delta, c) \text{ and } f \text{ is convex on } (c, c+\delta).$$

Note that in Definition 10.2, we assume tacitly that f is differentiable on $(c-\delta, c+\delta)\backslash\{c\}$. It is clear that if $f''(x) > 0$ for all $x \in (c-\delta, c)$ and $f''(x) < 0$ for all $x \in (c, c+\delta)$ for some δ or $f''(x) < 0$ for all $x \in (c-\delta, c)$ and $f''(x) > 0$ for all $x \in (c, c+\delta)$ then c is a point of inflection.

Necessary Condition for Point of Inflection

Theorem 10.2. *Let $c \in (a, b)$ and $f''(c)$ exist. If f has a point of inflection at c then $f''(c) = 0$.*

Proof (*). Assume that f' is strictly increasing on $(c-\delta, c)$ and is strictly decreasing on $(c, c+\delta)$ for some $\delta > 0$. Since $f''(c)$ exists,

$$f''(c) = \lim_{x \to c^-} \frac{f'(x) - f'(c)}{x - c} \geq 0.$$

Similarly $f''(c) = \lim_{x \to c^+} \frac{f'(x)-f'(c)}{x-c} \leq 0$. Therefore $f''(0) = 0$. □

Remark 10.2. It is possible that $f''(c) = 0$ at a point but c is not a point of inflection. For example, $f(x) = x^4$ and $c = 0$. It is also possible that $f''(c)$ may not exist but c could be a point of inflection. For example $f(x) = x^{1/3}$ and $c = 0$.

Asymptotes

We define three types of asymptotes called vertical asymptote, horizontal asymptote and slant asymptote for the graph of a given function. We can roughly say that a line is an asymptote of a graph if it comes as close as possible to that graph.

Vertical asymptote: Let $b \in \mathbb{R}$. The line $x = b$ is called a vertical asymptote for f if $\lim\limits_{x \to b^-} f(x) = \pm\infty$ or/and $\lim\limits_{x \to b^+} f(x) = \pm\infty$.

Horizontal asymptote: Let $c \in \mathbb{R}$. The line $y = c$ is called a horizontal asymptote for f if $\lim\limits_{x \to \infty} (f(x) - c) = 0$ or/and $\lim\limits_{x \to -\infty} (f(x) - c) = 0$.

Slant (or oblique) asymptote: A line defined by $y = mx + c$ where $m \neq 0$ and $c \in \mathbb{R}$, is called a slant asymptote for f if $\lim\limits_{x \to \infty} (f(x) - mx - c) = 0$ or/and $\lim\limits_{x \to -\infty} (f(x) - mx - c) = 0$.

In the preceding three definitions, it is understood that the function f is defined on a domain in such a way that the limits mentioned in the definitions are defined.

Example 10.3. 1. Let $f(x) = \frac{5x-2}{x-3}$. Then the line $x = 3$ is the vertical asymptote for f. Here, it is understood that the function f is defined on $\{x \in \mathbb{R} : x \neq 3\}$.

2. The function $f(x) = \frac{x^2-4}{x-2}$ has no vertical asymptotes, because, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x) = 4$.

3. Consider the function $f(x) = \frac{2x-4}{x-3}$. Then $f(x) = 2 + \frac{2}{x-3}$. Hence $y = 2$ is the horizontal asymptote for f.

4. Let $f(x) = \frac{x^2+x+1}{x-1}$. Then f has the vertical asymptote $x = 1$. Since $y = x + 2 + \frac{3}{x-1}$, the line $y = x + 2$ is the slant asymptote for f.

Curve sketching

For a given function f, the following tips help to roughly sketch the graph of f.

1. Domain: Find the domain of f. Determine the points of discontinuity of f, if any.

2. x and $y-$intercepts: If $f(0)$ exists, then the point $(0, f(0))$ is the $y-$intercept. If $f(x_0) = 0$ for some x_0, then $(x_0, 0)$ is an $x-$intercept. Find the x and $y-$intercepts of f, if exist and if possible.

3. Symmetry: Check whether the function is even, odd or neither. If f is even (i.e., $f(x) = f(-x)$ on the domain), then the graph is symmetric about the y-axis. If f is odd (i.e., $f(-x) = -f(x)$ on the domain), then the graph is (180° rotationally) symmetric about the origin.

4. Asymptotes: Determine the vertical, horizontal and slant asymptotes, if any.

5. Intervals of increasing and decreasing: Using f' and Application 7.2, determine the intervals where the function is increasing and decreasing.

6. Points of local maximum and minimum: Using the first or/and second derivative tests (Corollaries 10.1 and 10.2), find the points of local maximum and local minimum, if any.

7. Convexity/concavity and point of inflection. By solving the equation $f''(x) = 0$, find the candidates for the points of inflection. Using the first or second derivative of f determine the intervals where f is convex and concave. Determine the points of inflection, if any.

Example 9.4. We will sketch the graph of the function $f(x) = \frac{2x^3}{x^2-4}$.

1. The domain of the function is $\mathbb{R}\backslash\{-2, 2\}$.

2. Note that $(0, 0)$ is both x and $y-$intercept.

3. Since $f(x) = -f(-x)$, the graph is symmetric about the origin.

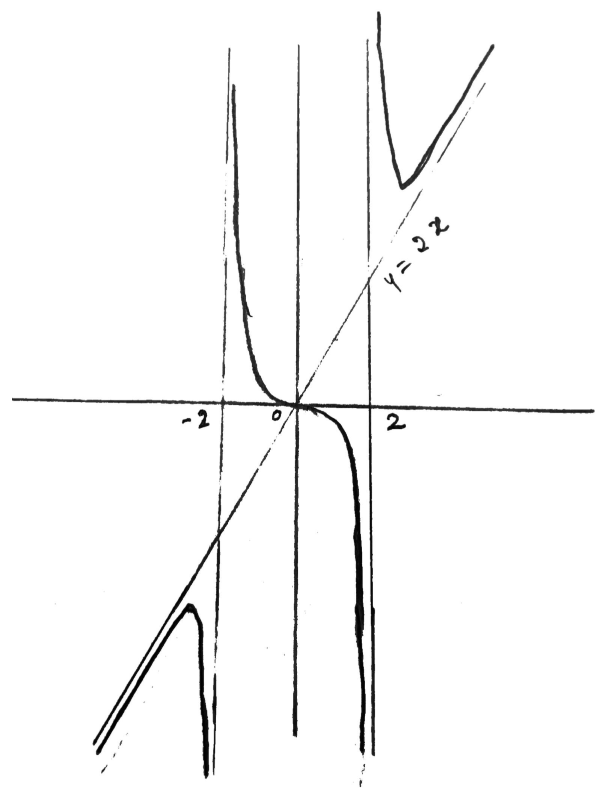
4. The vertical asymptotes are $x = 2$ and $x = -2$. Since $f(x) = 2x + \frac{8x}{x^2-4}$, $y = 2x$ is the slant asymptote.

5. Note that $f'(x) = \frac{2x^2(x^2-12)}{(x^2-4)^2}$. The function is f increasing on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$ as $f'(x) > 0$ in these intervals. The function is decreasing on $(-2\sqrt{3}, -2)$, $(-2, 2)$ and $(2, 2\sqrt{3})$ as $f'(x) < 0$ in these intervals

6. Since $f'(x) = 0$ at $x = -2\sqrt{3}, 0, 2\sqrt{3}$, these points are the candidates for the local maximum and local minimum. From the preceding observation, we conclude that $-2\sqrt{3}$ is the point of local maximum and $2\sqrt{3}$ is the point of local minimum.

7. Observe that $f''(x) = \frac{16x(x^2+12)}{(x^2-4)^3}$. Note that $f''(x) = 0$ at $x = 0$. Hence 0 is the candidate for the point of inflection. The function is convex on $(-2, 0)$ and $(2, \infty)$ as $f''(x) > 0$ in these intervals. The function is concave on $(-\infty, -2)$ and $(0, 2)$ as $f''(x) < 0$ in these intervals. Since f'' changes its sign at 0, 0 is the point of inflection.

A rough sketch of the graph of f is given below.


y = 2x
-2
0
2