Lectures 12: Infinite Series, Absolute convergence, Leibniz's test

It follows from Taylor's theorem that if $f : \mathbb{R} \to \mathbb{R}$ and $f^{(n+1)}$ exists on $\mathbb{R}$, then for any $x \neq 0$, there exists c between 0 and x such that

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

If we assume $f^{(n)}$ exists for all $n \in \mathbb{N}$, a natural question is whether we can write

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

for any $x \in \mathbb{R}$. For instance, can we write $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for any $x \in \mathbb{R}$? Observe that in the right hand side of the preceding equation, we add 1, $\frac{x}{1!}$, $\frac{x^2}{2!}$, $\frac{x^3}{3!}$, ... which is a sequence of real numbers. The first question is: how do we add a given sequence of real numbers (x_n)? For instance if $x_n = (-1)^{n-1}$ and if we add $x_1, x_2, ...$ in the following ways:

$$1 - 1 + 1 - 1 + \cdots = (1-1) + (1-1) + (1-1) + \cdots$$

and

$$1 - 1 + 1 - 1 + \cdots = 1 + (-1+1) + (-1+1) + (-1+1) + \cdots$$

we end up with different answers. So, first of all, we need to define the "sum" of a given sequence (x_n) of real numbers in a rigorous manner.

Let us start with a familiar sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$ We attempt to sum the sequence as follows. Define

$$S_1 = 1,\ S_2 = 1 + \frac{1}{2},\ S_3 = 1 + \frac{1}{2} + \frac{1}{4},\ S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8},\ S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, ...$$

Observe that $2 - S_1 = 1, 2 - S_2 = \frac{1}{2}, 2 - S_3 = \frac{1}{4}, 2 - S_4 = \frac{1}{8}, ...$ It is clear that no S_n can be equal to 2 but S_n tends to 2 as $n \to \infty$. In light of the above, there is a temptatation to say that

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

We will follow the process used above for summing up a sequence of real numbers.

Series and its convergence

Definition 12.1. Let (a_n) be a sequence of real numbers. Then an expression of the form $a_1 + a_2 + a_3 + \cdots$, denoted by $\sum_{n=1}^{\infty} a_n$, is called a series.

For each $n \in \mathbb{N}$, $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is called the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$.

If $S_n \to S$ for some S then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S. If (S_n) does not converge then, we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

If a series $\sum_{n=1}^{\infty} a_n$ converges to S, then we write $S = \sum_{n=1}^{\infty} a_n$.

Note that the notation $\sum_{n=1}^{\infty} a_n$ is used to denote the series $a_1 + a_2 + a_3 + \cdots$, as well as the sum of the series if it converges.

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Example 12.2. We use Definition 12.1 and test the convergence/divergence of some series.

1. The series $\sum_{n=1}^{\infty}(-1)^{n-1}$ diverges, because, (S_n) does not converge. We remark here that the given series can also be written as $\sum_{n=0}^{\infty}(-1)^n$. However, in both the cases S_n denotes the sum of the first n terms.

2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, because, $S_n = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \cdots + (\frac{1}{n}-\frac{1}{n+1}) = 1 - \frac{1}{n+1} \to 1$.

3. The series $\sum_{n=1}^{\infty} \ln(\frac{n+1}{n})$ diverges, because, $S_n = \ln(n+1)$ and (S_n) does not converge.

4. If $0 < |x| < 1$, then the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$, because, $S_{n+1} = 1 + x + \cdot\cdot \cdot + x^n = \frac{1-x^{n+1}}{1-x}$. Hence we write $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ if $|x| < 1$.

   In particular, $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 2$ which is already seen.

   Similarly, $0.99999.... = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots = \frac{9}{10}(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots) = 1$.

5. We now verify that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. By Example 3.2, the sequence (S_n) does not satisfy the Cauchy criterion. Hence (S_n) does not converge and therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Verifying the convergence/divergence of a given series using the definition is not an easy task. We will consider several tests which can be used for this purpose. We start with a simple result.

Necessary condition for convergence

Theorem 12.3. *If* $\sum_{n=1}^{\infty} a_n$ *converges then* $a_n \to 0$.

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Then $S_n \to S$ for some $S \in \mathbb{R}$. Observe that for every $n \in \mathbb{N}$, $a_{n+1} = S_{n+1} - S_n$. Since $S_n \to S$, we get $a_{n+1} \to S - S = 0$. □

The condition given in Theorem 12.3 is necessary but not sufficient i.e., it is possible that $a_n \to 0$ but $\sum_{n=1}^{\infty} a_n$ diverges. For example, by Example 12.2, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; however, $\frac{1}{n} \to 0$

Example 12.4. We now use Theorem 12.3 to test the divergence of certain series.

1. If $| x | \geq 1$, then the geometric series $\sum_{n=1}^{\infty} x^n$ diverges, because, $a_n \nrightarrow 0$.

2. The series $\sum_{n=1}^{\infty} \cos\frac{1}{n}$ diverges as $\cos\frac{1}{n} \to 1$.

3. If $p \leq 0$, then both $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$diverge.

4. Consider $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$. By the ratio test for sequence (Theorem 2.3), $\frac{n^3}{e^n} \to 0$. This implies that $\frac{e^n}{n^3} \to \infty$. Therefore, the given series diverges.

Absolute convergence

Definition 12.5. A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

The following result is useful.

Theorem 12.6. If a series converges absolutely then it converges.

Proof. Suppose $\sum_{n=1}^{\infty} | a_n |$ converges. Let (S_n) and $(\overline{S}_n)$ denote the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} | a_n |$ respectively. We will show that (S_n) satisfies the Cauchy criterion which proves the result. Note that for any $n > m$,

$$|S_n - S_m| = |\sum_{k=m+1}^{n} a_k| \ \leq \sum_{k=m+1}^{n} |a_k| = |\overline{S}_n - \overline{S}_m|.$$

Since $(\overline{S}_n)$ satisfies the Cauchy criterion, by the preceding inequality, (S_n) satisfies the Cauchy criterion. □

We need the following result to illustrate that a convergent series need not converge absolutely.

Theorem 12.7.(Leibniz test). *If (a_n) is decreasing and $a_n \to 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

Proof. Since (a_n) is decreasing and $a_n \geq 0$, we have

$$a_1 \geq a_1 - a_2 + a_3 \geq a_1 - a_2 + a_3 - a_4 + a_5 \geq ...$$

and

$$a_1 - a_2 \leq (a_1 - a_2) + (a_3 - a_4) \leq (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) \leq ....$$

This shows that (S_{2n+1}) is decreasing and (S_{2n}) is increasing. Now, for every n,

$$S_1 \geq S_{2n+1} = S_{2n} + a_{2n+1} \geq S_{2n} \geq S_2,$$

which implies that (S_{2n+1}) and (S_{2n}) are bounded. Hence (S_{2n+1}) and (S_{2n}) converge. Since $S_{2n+1} - S_{2n} = a_{2n+1} \to 0$, both (S_{2n+1}) and (S_{2n}) converge to the same limit. Therefore, (S_n) converges. □

The series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where $a_n \geq 0$ for all n, is called an alternating series.

Example 12.8. 1. By Theorem 12.7, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges. But, we have seen in Example 12.2 that it does not converge absolutely.

2. By Theorem 12.7, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ and $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln n}$ converge.

Elementary results

The first statement of the following result says that if we remove first few terms from a convergent (respectively divergent) series then the resulting series is also convergent (respectively divergent).

Theorem 12.9. 1. $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} a_{p+n}$ converges for any $p \geq 1$.

2. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and $\sum_{n=1}^{\infty} \lambda a_n$ converges for any $\lambda \in \mathbb{R}$.

Proof. We prove the first statement. The proof of the second statement follows similarly.

Let $p > 1$. Suppose (S_n) and $(\overline{S}_n)$ denote the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{n+p}$ respectively. Now, for any $n \in \mathbb{N}$, $\overline{S}_n = S_n - (a_1 + a_2 + \cdots + a_p)$. Observe that (S_n) converges if and only if $(\overline{S}_n)$ converges which proves the first statement. □

Example 12.10. 1. It follows from Theorem 12.9 that the series $\sum_{n=1}^{\infty}(\frac{1}{3^n} - \frac{1}{4^n})$ converges.

2. Consider the series $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{2^n})$. If this series converges, then by Theorem 12.9, the series $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{2^n} + \frac{1}{2^n})$ should converge which is not the case. Hence $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{2^n})$ diverges.

3. Consider the series $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$. The series is not an alternating series and hence Leibniz's test cannot be applied. However, note that the series is $\sum_{n=1}^{\infty}(a_n + b_n)$ where $a_n = \frac{(-1)^{n+1}}{2n-1}$ and $b_n = \frac{(-1)^{n+1}}{2n}$. By Leibniz's test both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and hence by Theorem 12.9, the given series converges.