Lecture 17: Riemann Integration (Part II)

In this lecture we will present some applications of Riemann criterion. We first present an example.

Example 17.1.(*) Let $f : [0,\ 1] \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ and } n > 1 \\ 0 & \text{otherwise.} \end{cases}$$

We will show that f is integrable and $\int_0^1 f(x)dx = 0$. We will use the Riemann criterion to show that f is integrable on $[0, 1]$.

Let $\varepsilon > 0$ be given. We will choose a partition P such that $U(P, f)-L(P, f) < \varepsilon$. Since $1/n \to 0$, there exists $N \in \mathbb{N}$ such that $1/n \in [0, \frac{\epsilon}{2}]$ for all $n > N$ and $\{\frac{1}{N}, \frac{1}{N-1}, ..., \frac{1}{3}, \frac{1}{2}\} \subset (\frac{\epsilon}{2}, 1)$. Find $x_N, y_N, x_{N-1}, y_{N-1}, ..., x_3, y_3, x_2, y_2$ such that $x_N < y_N < x_{N-1} < y_{N-1} < ... < x_3 < y_3 < x_2 < y_2$ and

$$\frac{1}{N} \in (x_N, y_N),\ \ \frac{1}{N-1} \in (x_{N-1}, y_{N-1}), ...,\ \ \frac{1}{2} \in (x_2, y_2)$$

and

$$|x_N - y_N| + |x_{N-1} - y_{N-1}| + ... + |x_2 - y_2| < \frac{\epsilon}{2}.$$

Consider the partition $P = \{0, \frac{\epsilon}{2}, x_N, y_N, x_{N-1}, y_{N-1}, ..., x_3, y_3, x_2, y_2, 1\}$. Observe that

$$U(P, f) = 1 \cdot \frac{\epsilon}{2} + 1 \cdot |x_N - y_N| + 1 \cdot |x_{N-1} - y_{N-1}| + ... + 1 \cdot |x_2 - y_2| < \epsilon$$

and $L(P, f) = 0$. Hence $U(P, f) - L(P, f) < \epsilon$. Therefore by the Reimann criterion f is integrable. Since the lower integral is 0 and the function is integrable, $\int_0^1 f(x)dx = 0$.

The following result which is a sequential version of the Riemann criterion is an immediate consequence of the Riemann criterion.

Theorem 17.1 (Riemann Criterion). Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable if and only if there exists a sequence (P_n) of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \to 0$.

Example 17.2. Let $f(x) = x^m$ for $x \in [a, b]$, $a \geq 0$ and $m \in \mathbb{N}$. We will use Theorem 17.1 and show that f is integrable. We will also use the argument involved in this example in the proof of Theorem 17.3. For $n \in \mathbb{N}$, choose a partition $P_n = \{a = x_0, x_1, x_2, ..., x_n = b\}$ such that $\Delta x_i = \frac{b-a}{n}$ for all $i = 1, 2, ..., n$. Observe that $M_i = x_i^m$ and $m_i = x_{i-1}^m$ for all $i = 1, 2, ..., n$. Hence

$$U(P_n, f) - L(P_n, f) = \sum_{n=1}^{n} (x_i^m - x_{i-1}^m)\frac{b-a}{n} = \frac{b-a}{n}(b^m - a^m) \to 0 \text{ as } n \to \infty$$

Therefore by Theorem 17.1, f is integrable.

We will apply the Riemann criterion to prove the following two existence theorems.

We need the following lemma.

Lemma 17.1. *Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be continuous. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$x, y \in [a, b] \text{ and } |x - y| < \delta\ \Rightarrow |f(x) - f(y)| < \epsilon. \tag{1}$$

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Proof.(*) Suppose that condition (1) does not hold. Then there exist an $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in $[a, b]$ such that $x_n - y_n \to 0$ and $| f(x_n) - f(y_n) | \geq \varepsilon_0$ for all $n \in N$. Since (x_n) is in $[a, b]$, by Theorem 4.1, there exists a subsequence (x_{ni}) of (x_n) such that $x_{ni} \to x_0 \in [a, b]$. Hence $y_{ni} \to x_0$. By continuity of f at x_0, it follows that $f(x_{ni}) \to f(x_0)$ and $f(y_{ni}) \to f(x_0)$. Therefore $| f(x_{ni}) - f(y_{ni}) | \to 0$. This contradicts the fact that $| f(x_{ni}) - f(y_{ni}) | \geq \varepsilon_0$ for all n_i. Hence condition (1) holds. □

Theorem 17.2. *If f is continuous on $[a, b]$ then f is integrable.*

Proof. (*) Let $\epsilon > 0$. Using Lemma 17.1, choose $\delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $x, y \in [a, b]$ and $|x - y| < \delta$.

Let $P = \{a = x_0, x_1, x_2, .., x_n = b\}$ be a partition of $[a, b]$ such that $\Delta x_i < \delta$ for all $i = 1, 2, \ldots, n$. Then, by Theorem 5.3, there exists $x_i^*, y_i^* \in [x_{i-1}, x_i]$ such that $f(x_i^*) = M_i$ and $f(y_i^*) = m_i$ for all $i = 1, 2, .., n$. Therefore, $M_i - m_i \leq \epsilon$ for all $i = 1, 2, ..., n$. Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \epsilon(b - a).$$

This implies that f is integrable. □

Theorem 17.3. *If f is a monotone function on $[a, b]$ then f integrable.*

Proof. Suppose f is monotonically increasing. For every $n \in \mathbb{N}$, choose a partition $P_n = \{a = x_0, x_1, x_2, .., x_n = b\}$ such that $\Delta x_i = \frac{b-a}{n}$ for all $i = 1, 2, ..., n$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ for all $i = 1, 2, ..., n$.. Therefore

$$\begin{aligned} U(P_n, f) - L(P_n, f) &= \frac{b-a}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= \frac{b-a}{n} [f(b) - f(a)] \end{aligned}$$

This shows that $U(P_n, f) - L(P_n, f) \to 0$ and hence by Theorem 17.1, f is integrable. The proof is similar in case f is decreasing. □

We need some properties of the integrals.

Properties of the integrals

Theorem 17.4. Let f and g be integrable on $[a, b]$.

1. If $c \in (a, b)$, then f is integrable on $[a, c]$ and $[c, d]$. Moreover, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$.
2. The function $f + g$ is integrable on $[a, b]$ and $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
3. For $\alpha \in \mathbb{R}$, the function αf is integrable and $\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx$.
4. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.
5. The function $|f|$, defined by $|f|(x) = |f(x)|$, is integrable and $| \int_a^b f(x)dx| \leq \int_a^b |f|(x)dx$.

We will not present the proof of Theorem 17.4 but we will use it.

We need the following natural convention.

Definition 17.1 Let f be integrable on $[a, b]$. Define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \quad \text{and} \quad \int_c^c f(x)dx = 0$$

for any $c \in \mathbb{R}$.